Math 595 - Additive Number Theory
Homework Problems, Fall 2008

1. (10 points) (a) Show that if $0 \le \alpha \le \beta \le 1$ and α is *rational*, then there is a set $\mathscr{A}$ of positive integers with Schnirelmann density α and asymptotic density β.

   (b) Determine, for which α, β with *irrational* α, the conclusion in (a) still holds.

2. (10 points) Find the Schnirelmann density of the set $\mathscr{A}$ of positive squarefree integers. You may use a computer to calculate $A(n)/n$ for "small" n.

3. (a) (5 points) Let $k \ge 2$, $p \ge 3$ prime, $a \ge 1$ and $s \ge \frac{p}{p-1}k$. Show that the congruence

$$x_1^k + \cdots + x_s^k \equiv m \pmod{p^a}$$

   is solvable with $(x_1, p) = 1$, for any m.

   (b) (10 points) Prove the following variant of the Cauchy-Davenport-Chowla theorem: Suppose $\mathscr{A}$ and $\mathscr{B}$ are sets of residues modulo n with $0 \in \mathscr{A}$, $0 \in \mathscr{B}$ and the only solution of $a + b \equiv 0 \pmod{n}$ with $a \in \mathscr{A}$, $b \in \mathscr{B}$ is $(a, b) = (0, 0)$. Show that

$$|\mathscr{A} + \mathscr{B}| \ge \min(n, |\mathscr{A}| + |\mathscr{B}| - 1).$$

4. (20 points) (a) Prove the following "average" version of Theorem 7: Let $\mathscr{B}$ be a basis, and for each m, let $h(m)$ be the smallest number of elements of $\mathscr{B}$ that have sum m. Define

$$h^* = \sup_{n \ge 1} \frac{1}{n} \sum_{m=1}^{n} h(m).$$

   Show that for any set $\mathscr{A}$,

$$\sigma(\mathscr{A} + \mathscr{B}) \ge \sigma(\mathscr{A}) + \frac{1}{2h^*}\sigma(\mathscr{A})(1 - \sigma(\mathscr{A})).$$

   (b) If $\mathscr{B} = S_2$, the set of perfect squares, show that $h^* = \frac{19}{6}$.

5. (10 points) Prove this analog of Lorentz's theorem for residues: Let $n \ge 2$ and $h \ge 2$. If $\mathscr{A} = \{a_1, \ldots, a_h\}$ is a set of incongruent residues modulo n, show that there is a complimentary set $\mathscr{B} = \{b_1, \ldots, b_k\}$ with every residue modulo n lying in $\mathscr{A} + \mathscr{B}$ and with

$$k \le Cn \frac{\log h}{h}.$$

   Work out an explicit value for the constant C.

6. (15 points) Prove that for every $h \geq 2$, there is an infinite B_h-set $\mathscr{A}$ such that

$$\limsup_{n\to\infty} \frac{A(n)}{n^{1/h}} \geq \left(\frac{1}{h}\right)^{1/h}.$$

7. (15 points) The *logarithmic density* of a set $\mathscr{A}$ is defined as

$$\delta(\mathscr{A}) = \lim_{n\to\infty} \frac{1}{\log n} \sum_{a\in\mathscr{A}, a\leq n} \frac{1}{a},$$

if the limit exists.

(i) Show that if the asymptotic density $d(\mathscr{A}) = \lim_{n\to\infty} \frac{1}{n} A(n)$ exists, then so does $\delta(\mathscr{A})$ and the two are equal.

(ii) Exhibit a set $\mathscr{A}$ such that $\delta(\mathscr{A})$ exists, but $d(\mathscr{A})$ does not.

8. (20 points) Show that

$$\sum_{n>x} \frac{1}{\phi^2(n)} \ll \frac{1}{x}$$

and

$$\sum_{n\leq x} \frac{1}{\phi(n)} \ll \log x.$$

That is, in the above sums, $\phi(n)$ "behaves like" cn for some $c > 0$. **Hint:** First show that for $k \in \{1, 2\}$,

$$\sum_{n\leq x} \left(\frac{n}{\phi(n)}\right)^k \ll x$$

by first writing

$$\left(\frac{n}{\phi(n)}\right)^k = \sum_{d|n} g(d).$$

9. (25 points) Fix $h \geq 4$. For $n \equiv h \pmod 2$, find and prove an asymptotic formula for the number of ways of writing n as a sum of h primes.

10. (25 points) Find and prove an asymptotic formula for the number of triples of primes $p_1, p_2, p_3 \leq x$ which form a 3-term arithmetic progression, i.e., $(p_3 - 2p_2 + p_1 = 0)$.

11. (15 points)

    (a) (5 pts) For each set, find an arithmetic progression $a \mod q$ that does not intersect it: (i) $8S_6$ (ii) $3S_3$. Thus show that $G(3) \geq 4$ and $G(6) \geq 9$.

    (b) (5 pts) Show that $G(k) \geq \frac{3k-1}{2}$ when $k = 3^m$ where $m \geq 1$ by finding an arithmetic progression $a \mod q$ that does not intersect $\frac{3k-3}{2} S_k$. Hint: induction on m.

    (c) (5 pts) Using congruences, show that $G(k) \geq 4k$ when $k = 2^m$ where $m \geq 2$. Hint: induction on m.

12. (10 points) Show that for $1 \leq j \leq k$ the jth iterate Δ_j of the forward difference operator satisfies

$$\begin{aligned}\Delta_j(x^k; h_1, \dots, h_j) &= \sum_{\substack{l_0, l_1, \dots, l_j \\ l_0 \geq 0, l_1 \geq 1, \dots, l_j \geq 1 \\ l_0 + l_1 + \cdots + l_j = k}} \frac{k!}{l_0! l_1! \cdots l_j!} x^{l_0} h_1^{l_1} \cdots h_j^{l_j} \\ &= h_1 \cdots h_j p_j(x; h_1, \dots, h_j),\end{aligned}$$

    where p_j is a polynomial in x of degree $k - j$ and having leading coefficient $\frac{k!}{(k-j)!}$.

13. (10 points) (a) Let $\tau_k(n)$ be the number of k-tuples of positive integers $(d_1, \dots, d_k)$ with $n = d_1 \cdots d_k$. Show that, for every k and every $\varepsilon > 0$, that $\tau_k(n) \ll_{k,\varepsilon} n^{\varepsilon}$.

    (b) Show that $\sum_{n \leq x} \tau_k(n) \ll x(\log x)^{k-1}$.