PART I. DUE MARCH 20

1. (10 points each) Use theorems from class to prove the following
   (a) Let $\Xi(x, y, z)$ denote the number of integers $n \leq x$ which have no prime factor in $(y, z]$. Prove that uniformly in $1.5 \leq y \leq z \leq x$, we have
   $$\Xi(x, y, z) \ll \frac{x \log y}{\log z}$$
   and uniformly for $1.5 \leq z \leq x/2$ that
   $$\Phi(x, z) = \Xi(x, 1.5, z) \gg \frac{x}{\log z}.$$  
   Lastly, prove the asymptotic
   $$\Phi(x, z) \sim e^{-\gamma} \frac{x}{\log z} \quad (z \to \infty, z = x^{o(1)}).$$
   In particular, this establishes Theorem Bu.
   (b) **Almost primes in short intervals.** Prove that for every $\delta > 0$, there is a natural number $k$ so that whenever $x \geq x_0(\delta)$, then the interval $(x, x + x^\delta]$ contains an integer $q$ with $\Omega(q) \leq k$.

2. (10 points) Use sieve ideas (e.g. the general sieve) to set up a *square-free sieve* to estimate
   $$\# \{n \leq x : n \text{ is square-free} \} = x \prod_p \left( 1 - \frac{1}{p^2} \right) + O(\sqrt{x}) = \frac{6}{\pi^2} x + O(\sqrt{x}).$$

3. (15 points) Suppose $F(x) \in \mathbb{Z}[x]$, $F = F_1 \cdots F_m$, where each $F_i$ has positive leading coefficient and is irreducible over $\mathbb{Q}$, no $F_i$ is a rational multiple of any other $F_j$, and for every prime $p$, there is an $n$ so that $p \nmid F(n)$. Prove that for some constant $C$, depending on $\deg(F)$, and for $x > x_0(F)$, $x_0(F)$ some constant depending on $F$, we have
   $$|\{n \leq x : \forall i, \Omega(F_i(n)) \leq C\}| \gg_F \frac{x}{\log^m x}.$$
4. (30 points) (a) In the lecture notes for the Brun-Hooley sieve, a convenient all-purpose choice was made for the parameters \( z_j \) and \( k_j \). Develop theorems analogous to Theorems BH.2 and BH.3 for more general choices of \( z_j \) and \( k_j \), keeping in mind the goals (U) and (L); e.g. the summation over the remainder terms should range up to a fixed power of \( z \), and the analog of the number \( E \) should be finite.

(b) it was shown in class that for infinitely many integers \( n \), \( \Omega(n) \leq 13 \) and \( \Omega(n + 2) \leq 13 \), and that for infinitely many primes \( p \), \( \Omega(p + 2) \leq 18 \). Using your theorems from part (a), improve the numbers 13 and 18.
PART II. DUE MAY 6 (incomplete list)

5. (20 points) Using “elementary” methods, show that
\[ \exp \left( c_1 \log x \log \log x \right) \leq \Psi(x, \log x) \leq \exp \left( c_2 \log x \log \log x \right) \]
for some constants 0 < c_1 < c_2 and large x.

6. (20 points) Prove that uniformly for 10 \leq \log^3 z \leq y \leq z \leq x,
\[ \# \left\{ n \leq x : \prod_{p \mid n, p \leq y} p^v > z \right\} = xe^{-u \log u + O(u \log \log(3u))}, \quad u = \frac{\log z}{\log y}. \]
That is, counting numbers which have a large y-smooth part.

7. (10 points) Show that for every \( \alpha < \frac{1}{2} \), a positive proportion of primes \( p \) satisfy both \( P^+(p-1) > p^\alpha \) and \( P^+(p+1) > p^\alpha \).

8. (10 points) (Selberg’s sieve) Let \( A_1, A_2, L, \kappa > 0 \), and let \( g \) be a multiplicative function satisfying
\[ (\Omega_1) \quad 0 \leq g(p) \leq 1 - A_1 \quad (p \text{ prime}) \]
and
\[ (\Omega_2(\kappa, L)) \quad -L \leq \sum_{w \leq p \leq y} g(p) \log p - \kappa \log \frac{y}{w} \leq A_2 \quad (2 \leq w \leq y). \]
Let \( h \) be the multiplicative function defined by \( h(p) = \frac{g(p)}{1-g(p)} \) for prime \( p \).
(i) Prove that uniformly for \( 2 \leq w \leq y \) and \( s \geq 0 \),
\[ \prod_{w \leq p \leq y} \left( 1 - \frac{1}{p^{s+1}} \right)^\kappa \left( 1 + \frac{h(p)}{p^s} \right) = 1 + O \left( \frac{L + 1}{\log w} \right). \]
Hint: be careful. You may want to consider separately the cases \( w < e^{L+1} \) and \( w \geq e^{L+1} \).
(ii) Use (i) to show that
\[ \lim_{s \to 0^+} \prod_p \left( 1 - \frac{1}{p^{s+1}} \right)^\kappa \left( 1 + \frac{h(p)}{p^s} \right) = \prod_p \left( 1 - \frac{1}{p} \right)^\kappa (1 + h(p)). \]