Simultaneous Distribution of the Fractional Parts of Riemann Zeta Zeros

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Abstract

In this paper, we investigate the simultaneous distribution of the fractional parts of \( \{ \alpha_1 \gamma, \alpha_2 \gamma, \cdots, \alpha_n \gamma \} \), where \( n \geq 2 \), \( \alpha_1, \alpha_2, \cdots \), and \( \alpha_n \) are fixed distinct positive real numbers and \( \gamma \) runs over the imaginary parts of the non-trivial zeros of the Riemann zeta-function.

1 Introduction and statement of results

Let \( \alpha \) be a fixed positive real number, and \( \gamma \) run over the imaginary parts of the zeros of the Riemann zeta-function. We are interested in the distribution of the fractional parts \( \{ \alpha \gamma \} \). Rademacher [8] was the first to consider this problem and he conjectured that, for a certain specific type of \( \alpha \), there should be a "predominance of terms which fulfill \(|\{\alpha \gamma\} - 1/2| < 1/4". Since the fractional parts are uniformly distributed modulo 1, as proved by Hlawka [5] in 1975, any discrepancy must be very subtle. The first and third authors uncovered in [2] this delicate inequity in the fractional parts and not only proved Rademacher correct, but also gave a much more precise measure for this phenomenon.

Let \( T = \mathbb{R}/\mathbb{Z} \) be the torus. Since the fractional parts \( \{\alpha \gamma\} \) are uniformly distributed (mod 1) for any fixed \( \alpha \), for all continuous functions \( h : T \to \mathbb{C} \), we have

\[
\sum_{0 < \gamma \leq T} h(\alpha \gamma) = N(T) \int_T h(u)du + o(N(T)),
\]

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as \( T \to \infty \), where \( N(T) \) denotes the number of zeros \( 0 < \gamma \leq T \). We know that ([10], Theorem 9.4)
\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{T}{2\pi} + O(\log T), \quad (T \geq 1).
\]

The first and third authors [2] showed that, for a large class of functions \( h : \mathbb{T} \to \mathbb{C} \), as \( T \to \infty \),
\[
\frac{1}{T} \sum_{0 < \gamma \leq T} h(\alpha \gamma) - \frac{N(T)}{T} \int_{\mathbb{T}} h(u) du = \int_{\mathbb{T}} h(u) g_\alpha(u) du + o(1),
\]
where \( g_\alpha(u) \) is a function depending on the form of \( \alpha \). From this result, we can see that the right hand side is close to a constant, and thus the discrepancy of the set \( \{h(\alpha \gamma) : 0 < \gamma \leq T\} \) is of order \( O(\frac{1}{\log T}) \). In [3], the first author, Soundararajan, and the third author established connections between the discrepancy of this set, Montgomery’s pair correlation function and the distribution of primes in short intervals.

In the present paper, our goal is to generalize the results from [2] to the case of simultaneous distribution of fractional parts \( \{\alpha_1 \gamma\}, \{\alpha_2 \gamma\}, \ldots, \{\alpha_n \gamma\} \), where \( \alpha_1, \ldots, \alpha_n \) are distinct, positive real numbers. As we will see below, a new phenomenon involving Diophantine approximation appears in the higher dimensional case \( n \geq 2 \).

First, we consider the case \( n = 2 \). Let \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \), \( \alpha_1, \alpha_2 > 0 \). Let \( h(x, y) \) be a function defined on the two dimensional torus \( \mathbb{T}^2 \). Assume \( h \) has Fourier expansion
\[
\sum_{m,l} c_{m,l} e^{2\pi i (mx + ly)},
\]
where
\[
c_{m,l} \ll \left( \frac{1}{\max\{|m|, |l|\} + 1} \right)^A, \quad (1.1)
\]
for some \( A > 4 \).

Next, we define a density function \( g_\alpha \). If there is no \( (m, l) \in \mathbb{Z}^2 \) with \( \gcd(m, l) = 1 \) such that \( m\alpha_1 + l\alpha_2 \) is of the form \( \frac{a \log p}{q} \frac{2\pi}{q} \) for some positive integers \( a, q \) and prime \( p \), then, for such \( \alpha = (\alpha_1, \alpha_2) \), we let
\[
g_\alpha(x, y) = 0, \quad (x, y \in \mathbb{R}). \quad (1.2)
\]
If there is a unique pair \((m_0, l_0)\) with \(\gcd(m_0, l_0) = 1\) such that
\[
m_0\alpha_1 + l_0\alpha_2 = \frac{a \log p}{q} 2\pi\tag{1.3}
\]
for some positive integers \(a, q\) and prime \(p\), then, for such \(\alpha = (\alpha_1, \alpha_2)\), let
\[
g_\alpha(x, y) = -\frac{1}{\pi} (\log p) \Re \left\{ \sum_{k \geq 1} p^{-\frac{a_k}{2}} e^{-2kq_1\pi i (m_0x + l_0y)} \right\}. \tag{1.4}
\]

If there are two pairs \((m_1, l_1) \neq (m_2, l_2)\) with \(\gcd(m_1, l_1) = 1\) and \(\gcd(m_2, l_2) = 1\) such that
\[
m_1\alpha_1 + l_1\alpha_2 = \frac{a_1 \log p_1}{q_1} 2\pi, \quad \text{and} \quad m_2\alpha_1 + l_2\alpha_2 = \frac{a_2 \log p_2}{q_2} 2\pi, \tag{1.5}
\]
for some positive integers \(a_1, a_2, q_1, q_2\) and primes \(p_1 \neq p_2\), then these two linear combinations are linearly independent, and for such \(\alpha = (\alpha_1, \alpha_2)\), let
\[
g_\alpha(x, y) = -\frac{1}{\pi} (\log p_1) \Re \left\{ \sum_{k \geq 1} p_1^{-\frac{a_1k}{2}} e^{-2kq_1\pi i (m_1x + l_1y)} \right\} - \frac{1}{\pi} (\log p_2) \Re \left\{ \sum_{k \geq 1} p_2^{-\frac{a_2k}{2}} e^{-2kq_2\pi i (m_2x + l_2y)} \right\}. \tag{1.6}
\]

We prove the following theorems.

**Theorem 1** Assume a function \(h(x, y)\) defined on \(\mathbb{T}^2\) satisfies (1.1). Then, if \(\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2\) satisfies
\[
|m\alpha_1 + l\alpha_2| > \frac{C}{e_{\max\{|m|, |l|\}}}
\]
for all \((m, l) \in \mathbb{Z}^2\) and some constant \(C\), we have
\[
\lim_{T \to \infty} \frac{1}{T} \left( \sum_{0 < \gamma \leq T} h(\alpha_1\gamma, \alpha_2\gamma) - N(T) \int_{\mathbb{T}^2} h \right) = \int_{\mathbb{T}^2} hg_\alpha,
\]
where \(g_\alpha\) is defined as above.

**Remark.** By a theorem of Khintchine [7], the set of such \(\alpha\) has full Lebesgue measure.
Theorem 2 Assume there are two pairs \((m_1, l_1) \neq (m_2, l_2)\) with \(\gcd(m_1, l_1) = 1\) and \(\gcd(m_2, l_2) = 1\) satisfying (1.5) for some positive integers \(a_1, a_2, q_1, q_2\) and primes \(p_1 \neq p_2\). Then for such \(\alpha = (\alpha_1, \alpha_2)\), we have
\[
\lim_{T \to \infty} \frac{1}{T} \left( \sum_{0 < \gamma \leq T} h(\alpha_1 \gamma, \alpha_2 \gamma) - N(T) \int_{T^2} h \right) = \int_{T^2} h g_\alpha.
\]

Actually, we can prove that for any pair \(\alpha = (\alpha_1, \alpha_2)\), there is an infinite sequence of “long” intervals of \(T\)-values on which we may draw the conclusion of the previous theorem. Consider the convergents \(\frac{p_n}{q_n}\) of the continued fraction of \(\xi := \frac{\alpha_1}{\alpha_2}\).

Theorem 3 Assume (1.1) and fix any \(\epsilon > 0\). Let \(U_\alpha = \bigcup_{n=1}^{\infty} [q_n^{1+\epsilon}, e^{q_n^{A+\epsilon}}] \). Then,
\[
\lim_{T \to \infty} \frac{1}{T} \left( \sum_{0 < \gamma \leq T} h(\alpha_1 \gamma, \alpha_2 \gamma) - N(T) \int_{T^2} h \right) = \int_{T^2} h g_\alpha.
\]

For the higher dimensional case when \(n \geq 3\), consider a function \(h(x)\) defined on \(\mathbb{T}^n\), which has Fourier expansion
\[
h(x) = \sum_m c_m e^{2\pi i m \cdot x},
\]
and assume that
\[
|c_m| \ll \left( \frac{1}{\|m\| + 1} \right)^B,
\]
where \(B > n + 2\) is a constant, and \(\| \cdot \|\) is the sup-norm.

Let \(\alpha \in \mathbb{R}^n\). If there exists an \(r \times n\) \((r \leq n)\) matrix \(M = (b_{ij})\) with integer entries, \(\text{Rank}(M) = r\), and \(\gcd(b_{i,1}, b_{i,2}, \cdots, b_{i,n}) = 1\) for all \(1 \leq i \leq r\) such that
\[
M \alpha^\top = P,
\]
where \(P = (\frac{a_1 \log p_1}{2\pi}, \cdots, \frac{a_r \log p_r}{2\pi})^\top\) for some integers \(a_i, q_i\), and distinct primes \(p_i, (i = 1, \cdots, r)\), we define
\[
g_\alpha(x) = -\frac{1}{\pi} \sum_{j=1}^{r} (\log p_j) \Re \left\{ \sum_{k \geq 1} p_j^{-\frac{a_j}{2}} e^{-2kq_j \pi i (b_j \cdot x)} \right\}
= -\frac{1}{\pi} \sum_{j=1}^{r} \frac{(\log p_j)(p_j^{-\frac{a_j}{2}} \cos(2\pi q_j (b_j \cdot x) - 1) - 2p_j^{-\frac{a_j}{2}} \cos(2\pi q_j (b_j \cdot x) + 1)}{p_j^{a_j} - 2p_j^{a_j} \cos(2\pi q_j (b_j \cdot x) + 1)}.
\]
where \( b_j = (b_{j,1}, b_{j,2}, \cdots, b_{j,n}) \) for \( 1 \leq j \leq r \). If there is no such matrix, we define
\[
g_\alpha(x) = 0, \tag{1.10}
\]
for all \( x \in \mathbb{T}^n \). Then, we have the following.

**Theorem 4** Assume \( h(x) \) satisfies condition (1.7). Then, if \( \alpha \in \mathbb{R}^n \) satisfies
\[
|m \cdot \alpha| > \frac{C}{e^{||m||}} \tag{1.11}
\]
for all \( m \in \mathbb{Z}^n \) and some fixed constant \( C \), we have
\[
\lim_{T \to \infty} \frac{1}{T} \left( \sum_{0 < \gamma \leq T} h(\gamma \alpha) - N(T) \int_{\mathbb{T}^n} h \right) = \int_{\mathbb{T}^n} h g_\alpha.
\]

For the special case when \( r = n \) and \( \text{Rank}(M) = n \), we prove the following result.

**Theorem 5** If \( h(x) \) satisfies condition (1.7) and \( \alpha \) satisfies condition (1.8) for \( r = n \), we have
\[
\lim_{T \to \infty} \frac{1}{T} \left( \sum_{0 < \gamma \leq T} h(\gamma \alpha) - N(T) \int_{\mathbb{T}^n} h \right) = \int_{\mathbb{T}^n} h g_\alpha.
\]

Finally, we show some graphs comparing the limiting density function \( g_\alpha \) with numerical computations obtained from the first 100 million zeros of the Riemann zeta-function (which were kindly provided by Tomás Oliveira e Silva, see [9]). Define
\[
M(y_1, y_2; T) = M_\alpha(y_1, y_2; T) := \frac{1}{T} \sum_{0 < \gamma \leq T, \{\alpha_1\gamma\} < y_1, \{\alpha_2\gamma\} < y_2} 1 - y_1 y_2 \frac{N(T)}{T}.
\]

Take \( T = 42653549.761 \), then \( N(T) = 10^8 \). Denote \( \Delta = \frac{1}{100} \). We partition \([0,1) \times [0,1)\) into \( \Delta^{-2} \) small rectangles with side length \( \Delta \). For some values of \( \alpha_1, \alpha_2 \), we plot the values of \( DM := \frac{1}{\Delta^2} (M(y_1 + \Delta, y_2 + \Delta) - M(y_1 + \Delta, y_2) - M(y_1, y_2 + \Delta) + M(y_1, y_2)) \) for each small rectangle \([y_1, y_1 + \Delta) \times [y_2, y_2 + \Delta)\).

**Example 1.** Let \( \alpha_1 \) and \( \alpha_2 \) satisfy
\[
\alpha_1 + \alpha_2 = \frac{\log 2}{2\pi},
\]
and
\[
\alpha_1 \alpha_2 = \frac{\log 2}{4\pi^2}.
\]
\[ \alpha_1 - \alpha_2 = \frac{\log 3}{4\pi}. \]

Figure 1.1 shows the graph of \( g_{\alpha} \). Figure 1.2 shows the values of \( DM \), which are a close approximation of \( g_{\alpha} \).

**Example 2.** Figure 1.3 and Figure 1.4 show the analogous graphs for the case when \( \alpha_1 \) and \( \alpha_2 \) satisfy

\[
\begin{align*}
2\alpha_1 + \alpha_2 &= \frac{\log 5}{2\pi}, \\
2\alpha_1 + 3\alpha_2 &= \frac{\log 7}{2\pi}.
\end{align*}
\]
Figure 1.2: Values of $DM$

Figure 1.3: Graph of $g_{\alpha}$
2 Proof of Theorems 1-3

We may assume that $\alpha_1/\alpha_2$ is positive and irrational, else the problem reduces to the $n = 1$ case. We need Lemma 1 from [2], which is an extension of a famous formula of Landau. Let $\rho$ denote a generic nontrivial zero of the Riemann zeta-function, and denote $\gamma = \Im \rho$.

**Lemma 1** Let $x, T > 1$, and denote by $n_x$ the nearest prime power to $x$. Then

$$\sum_{0 < \gamma \leq T} x^\rho = -\frac{\Lambda(n_x)}{2\pi} e^{iT \log(x/n_x)} - 1 + O\left( x \log^2(2xT) + \frac{\log 2T}{\log x} \right),$$

where if $x = n_x$ the first term is $-T \frac{\Lambda(n_x)}{2\pi}$.

**Proof of Theorems 1, and 3.** Assume a function $h(x, y)$ is defined on the two dimensional torus and has Fourier series expansion given by

$$h(x, y) = \sum_{m, l} c_{m, l} e^{2\pi i(mx + ly)} = \sum_{|m|, |l| \leq J} c_{m, l} e^{2\pi i(mx + ly)} + O\left( \sum_{\text{max}(|m|, |l|) > J} |c_{m, l}| \right), \quad (2.1)$$

where $J \leq \log T$ is a parameter to be chosen later.
By our assumption (1.1), one can see that the error term above is \(O(1/A^2)\). Thus, by (2.1), we get

\[
\sum_{0<\gamma \leq T} h(\alpha_1\gamma, \alpha_2\gamma) = \sum_{0<\gamma \leq T} \left( \sum_{|m|,|l| \leq J} c_{m,l} e^{2\pi i (m\alpha_1 + l\alpha_2)\gamma} \right) + O\left(\frac{N(T)}{J^2}\right).
\]

The second equality is a consequence of the identity \(c_{-m,-l} = \overline{c_{m,l}}\). Let \(x_{m,l} = e^{2\pi i (m\alpha_1 + l\alpha_2)}\). By the proof of (3.8) in [2], for \(1 < x \leq \exp\left\{\log T / 50 \log \log T\right\}\), we have

\[
\sum_{0<\gamma \leq T} (x_i\gamma - x^\rho - 1/2) \ll \frac{T \log^2 x}{\log T} + \frac{T}{\log 10 T}.
\]

We now break the double sum into two pieces, over those pairs \((m,l)\) corresponding to “large” \(x_{m,l}\) and those pairs \((m,l)\) corresponding to “small” \(x_{m,l}\). Fix an arbitrary positive constant \(C\) and denote

\[
E_j := \left\{ (m,l) \in \mathbb{Z}^2 : 0 < \max\{|m|,|l|\} \leq J, m\alpha_1 + l\alpha_2 > \min\left(\frac{C}{e^{\max\{|m|,|l|\}}}, \frac{|\alpha_2|}{2m}, \frac{1}{4\pi}\right) \right\},
\]

\[
F_j := \left\{ (m,l) \in \mathbb{Z}^2 : 0 < \max\{|m|,|l|\} \leq J, 0 < m\alpha_1 + l\alpha_2 \leq \min\left(\frac{C}{e^{\max\{|m|,|l|\}}}, \frac{|\alpha_2|}{2m}, \frac{1}{4\pi}\right) \right\}.
\]

Write

\[
I_E = \sum_{(m,l) \in E_j} c_{m,l} \sum_{0<\gamma \leq T} x_{m,l}^{\rho-1/2}, \quad I_F = \sum_{(m,l) \in F_j} c_{m,l} \sum_{0<\gamma \leq T} x_{m,l}^{\rho-1/2}.
\]

Applying Lemma 1 and using \(A > 4\), we get

\[
I_E = -\frac{1}{2\pi} \sum_{(m,l) \in E_j} \frac{c_{m,l} \Lambda(n_{x_{m,l}})}{\sqrt{x_{m,l}}} \frac{\sin(T \log \frac{x_{m,l}}{n_{x_{m,l}}})}{\log \frac{x_{m,l}}{n_{x_{m,l}}}}.
\]
\[ + O \left( \sum_{(m,l) \in E_J} |c_{m,l}| \left( \sqrt{x_{m,l}} \log^2 (2x_{m,l}T) + \frac{\log 2T}{\sqrt{x_{m,l}}} \right) \right) \]

\[ = - \frac{1}{2\pi} \sum_{(m,l) \in E_J} \frac{c_{m,l} \Lambda(n_{x_{m,l}})}{\sqrt{x_{m,l}}} \sin \left( T \log \frac{x_{m,l}}{n_{x_{m,l}}} \right) \log \frac{x_{m,l}}{n_{x_{m,l}}} + O \left( e^{O(J)} \log^2 T \right). \quad (2.4) \]

Here we have used the fact that for \((m,\ell) \in E_J,\)

\[ \log x_{m,\ell} = 2\pi (m\alpha_1 + l\alpha_2) \geq e^{-J}. \]

By standard facts on Diophantine approximation, we know that if \((m, l) \in F_J\) then

\[ \left| \frac{\alpha_1}{\alpha_2} + \frac{l}{m} \right| < \frac{1}{2m^2}, \]

and hence all elements of \(F_J\) are of the form \((q_n, -p_n),\) where \(p_n/q_n\) is a convergent of the continued fraction of \(\frac{\alpha_1}{\alpha_2}.\) As \(q_n\) grows at least exponentially and \(q_n \leq J\) for \((q_n, -p_n) \in F_J,\) there are \(O(\log J) = O(\log \log T)\) elements in \(F_J.\) Let \(n^*\) denote the largest \(n\) such that \((q_n, -p_n) \in F_J.\)

By elementary properties of continued fractions (Chapter X, [4], or Corollary 1.4, [1]), we have

\[ \frac{\alpha_2}{q_j + q_{j+1}} < |q_j \alpha_1 - p_j \alpha_2| < \frac{\alpha_2}{q_{j+1}}. \quad (2.5) \]

Thus, for all \(j\) except possibly \(j = n^*,\) we have \(q_{j+1} \leq J\) and from Lemma 1 we get (writing \(x_j = x_{q_j, -p_j}\))

\[ \sum_{j \neq n^*} c_{q_j, -p_j} \sum_{0 < \gamma \leq T} x_j^{p-1/2} = -\frac{1}{2\pi} \sum_{j \neq n^*} c_{q_j, -p_j} \Lambda(n_{x_j}) \frac{\sin(T \log \frac{x_j}{n_{x_j}})}{\sqrt{x_j} \log \frac{x_j}{n_{x_j}}} + O(\log^3 T) \quad (2.6) \]

where we have used that \(n_{x_j} = 2.\) We adopt the convention that the term \(j = n^*\) is included in the sum above if \(q_{n^*+1} \leq J.\) In fact, the error term from Lemma 1 is acceptable, namely \(o(T),\) if \(q_{n^*+1} = o(T/\log T)\) as well.

Now let \(J = \sqrt{\log T}\) and assume that \((\log T)^{1/4} \log \log T < q_{n^*} \leq J.\) We revert to the sum over \(x_{q_{n^*}, -p_{n^*}}.\) By (2.2) and the the trivial bound for \(\sum_{0 < \gamma \leq T} x_{q_{n^*}, -p_{n^*}}^{p-1/2},\) we have

\[ c_{q_{n^*}, -p_{n^*}} \sum_{0 < \gamma \leq T} x_{q_{n^*}, -p_{n^*}}^{p-1/2} \ll \frac{T^{1/4}}{q_{n^*}^{1/4}} = o(T). \]
Combined with (2.4) and (2.6), we see that if (a) \( F_j \) is empty or if (b) there is some convergent of \( \alpha_1/\alpha_2 \) with \( q_n \in ((\log T)^{1/4} \log \log T, o(T/ \log T)] \), then

\[
\sum_{0 < \gamma \leq T} h(\alpha_1\gamma, \alpha_2\gamma) = -\frac{T}{\pi} \Re \sum_{\substack{\max\{m,|l|\} \leq J \atop m\alpha_1 + l\alpha_2 > 0}} \frac{c_{m,l}\Lambda(n_{x,m,l}) \sin(T \log \frac{x_{m,l}}{n_{x,m,l}})}{\sqrt{x_{m,l}}} - \frac{T}{\pi} \Re \sum_{\substack{\max\{m,|l|\} \leq J \atop m\alpha_1 + l\alpha_2 > 0}} \frac{c_{m,l}\Lambda(n_{x,m,l}) \sin(T \log \frac{x_{m,l}}{n_{x,m,l}})}{\sqrt{x_{m,l}}} + N(T) \int h + o(T). \tag{2.7}
\]

Note that (b) is satisfied if \( T \) is large enough and \( T \in \bigcup_{n=1}^{\infty} [q_n^{1+\epsilon}, e^{\pi n^{A-\epsilon}}] \).

For the main term, \( \frac{\Lambda(n_{x,m,l})}{\sqrt{x_{m,l}}} \) is bounded, and hence the sum is absolutely and uniformly convergent in \( T \) under our assumption (1.1). Also every term with \( n_{x,m,l} \neq x_{m,l} \) tends to zero as \( T \to \infty \). Thus, by (2.7), as \( T \to \infty \),

\[
\frac{1}{T} \left( \sum_{0 < \gamma \leq T} h(\alpha_1\gamma, \alpha_2\gamma) - N(T) \int h \right) = -\frac{1}{\pi} \Re \left\{ \sum_{\substack{\max\{m,|l|\} \leq J \atop m\alpha_1 + l\alpha_2 > 0}} \frac{c_{m,l}\Lambda(x_{m,l})}{\sqrt{x_{m,l}}} \right\} + o(1). \tag{2.8}
\]

Since \( \Lambda(x_{m,l}) = 0 \) unless \( x_{m,l} = e^{2\pi (m\alpha_1 + l\alpha_2)} \) is a prime power, by (1.2), when \( g_{\alpha} = 0 \), the sum in the right hand side of (2.8) is also 0. If \( \alpha \) satisfies (1.3), as \( T \to \infty \), then the right hand side of (2.8) becomes

\[
-\frac{1}{\pi} \Re \left\{ \sum_{k \geq 1} (\log p)p\frac{-a_k}{\pi} c_{kq_m,kq_l} \right\},
\]

which equals \( \int h g_{\alpha} \) by (1.4). If \( \alpha \) satisfies (1.5), as \( T \to \infty \), the right hand side of (2.8) becomes

\[
-\frac{1}{\pi} \Re \left\{ \sum_{k \geq 1} (\log p_1)p_1\frac{-a_{k_1}}{\pi} c_{kq_{m_1},kq_{l_1}} \right\} - \frac{1}{\pi} \Re \left\{ \sum_{k \geq 1} (\log p_2)p_2\frac{-a_{k_2}}{\pi} c_{kq_{m_2},kq_{l_2}} \right\},
\]

which equals \( \int h g_{\alpha} \) by (1.6). So in all cases, we have

\[
\lim_{T \to \infty} \frac{1}{T} \left( \sum_{0 < \gamma \leq T} h(\alpha_1\gamma, \alpha_2\gamma) - N(T) \int h \right) = \int h g_{\alpha}.
\]

Thus, Theorems 1 and 3 follow.
Proof of Theorem 2. Solving the two linear equations, we obtain

\[
\alpha_1 = \frac{a_{11}}{2\pi} \log p_1 + \frac{a_{12}}{2\pi} \log p_2, \quad \text{and} \quad \alpha_2 = \frac{a_{21}}{2\pi} \log p_1 + \frac{a_{22}}{2\pi} \log p_2,
\]

where the numbers \(a_{ij}\) are rational numbers depending on \(a_1, a_2, q_1, q_2\). By Baker’s theorem (See [11], Chapters 4 and 6), there exists a number \(\mu\) such that for all \((m, l) \in \mathbb{Z}^2\),

\[
|m\alpha_1 + l\alpha_2| > \frac{D}{(\max\{|m|, |l|\} + 1)^\mu},
\]

for some constant \(D\) depending on \(\alpha\). Thus, for appropriate \(C\) the set \(F_J\) above will be empty and Theorem 2 follows from Theorem 1.

Remarks. Suppose that there is a very long gap between convergents, say \(q_n^*\) is small and \(q_{n+1}^*\) is very large. Put \(x = x_{q_n^*, -p_n^*}\) and assume that \(\log x \ll \frac{\log T}{T}\). Using the known estimate \(N(T) = \frac{T}{2\pi} \log(\frac{T}{2\pi e}) + O(\log T)\), we get

\[
\sum_{0 < \gamma \leq T} x^{i\gamma} = \int_0^T x^{it} \log \left( \frac{t}{2\pi} \right) dt + O(\log^2 T)
\]

\[
= T \log(T/2\pi) \frac{e^{iT\log x} - 1}{iT \log x} + O(T).
\]

In the range \(T \ll 1/\log x\), there is thus a term in the sum (2.3) of order \(N(T)|c_{q_n^*, -p_n^*}|\), which may be larger than \(T\).

3 Higher dimensional case

In this section, we present the proof of Theorem 4 and Theorem 5.

Proof of Theorem 4. By Theorem 2 in [6], we know that the set of \(\alpha\)'s under the condition (1.11) has full Lebesgue measure.

Assume a function \(h(x)\) defined on \(\mathbb{T}^n\) has Fourier expansion given by

\[
h(x) = \sum_m c_m e^{2\pi i (m \cdot \alpha)} = \sum_{0 < |m| \leq J} c_m e^{2\pi i (m \cdot \alpha)} + O \left( \sum_{|m| > J} |c_m| \right),
\]

where \(J \leq \log T\) is a parameter to be chosen later.
By our assumption (1.7), one can see that the error term above is $O(1/J^{B-n})$. Thus, by (3.1), we have

$$\sum_{0<\gamma \leq T} h(\gamma \alpha) = \sum_{0<\gamma \leq T} \left( \sum_{||m|| \leq J} c_m e^{2\pi i \gamma (m \cdot \alpha)} \right) + O(\frac{N(T)}{J^{B-n}})$$

$$= N(T) \int h + 2\Re \sum_{0<||m|| \leq J} c_m \sum_{0<\gamma \leq T} \frac{e^{2\pi i \gamma (m \cdot \alpha)}}{N(T) J^{B-n}}.$$

The second equality is a consequence of the identity $c_{-m} = \bar{c}_m$. Let $x_m = e^{2\pi (m \cdot \alpha)}$. Hence, from (1.7) and (2.2), we have for $T$ large and $J \ll (\log \log T)^2$ that

$$\sum_{0<\gamma \leq T} h(\gamma \alpha) - N(T) \int h = 2\Re \sum_{0<||m|| \leq J} c_m \sum_{0<\gamma \leq T} x_m^{\rho-1/2} + O \left( \frac{N(T)}{J^{B-n}} + \frac{T}{\log T} \right). \quad (3.2)$$

Write

$$I = \sum_{0<||m|| \leq J} c_m \sum_{0<\gamma \leq T} x_m^{\rho-1/2}.$$

Applying Lemma 1, we get

$$I = -\frac{1}{2\pi} \sum_{m \cdot \alpha > 0, 0<||m|| \leq J} \frac{c_m \Lambda(n x_m)}{\sqrt{x_m}} \frac{\sin(T \log \frac{x_m}{n x_m})}{\log \frac{x_m}{n x_m}}$$

$$+ O \left( \sum_{m \cdot \alpha > 0, 0<||m|| \leq J} |c_m| \left( \sqrt{x_m} \log^2(2x_m T) + \frac{\log 2T}{\sqrt{x_m \log x_m}} \right) \right)$$

$$= -\frac{1}{2\pi} \sum_{m \cdot \alpha > 0, 0<||m|| \leq J} \frac{c_m \Lambda(n x_m)}{\sqrt{x_m}} \frac{\sin(T \log \frac{x_m}{n x_m})}{\log \frac{x_m}{n x_m}} + O \left( e^{O(J)} \log^2 T \right). \quad (3.3)$$

Here we used our assumption (1.11) that

$$\log x_m = 2\pi (m \cdot \alpha) \geq \frac{2\pi C}{e^T}.$$
Now let $J = \sqrt{\log T}$. By (3.2), (3.3) and $B > n + 2$, we get

$$
\sum_{0 < \gamma \leq T} h(\gamma \alpha) - N(T) \int h = \frac{T}{\pi} \sum_{\substack{|m\alpha| > 0 \\|m\| \leq J}} \frac{c_m \Lambda(n_{x_m}) \sin(T \log \frac{x_m}{n_{x_m}})}{\sqrt{x_m} \log \frac{x_m}{n_{x_m}}} + o(T). \quad (3.4)
$$

Since $\frac{\Lambda(n_{x_m})}{\sqrt{x_m}}$ is bounded, the first sum is absolutely and uniformly convergent in $T$ under the assumption (1.7). And the terms with $n_{x_m} \neq x_m$ tend to zero as $T \to \infty$. Thus, by (3.4), as $T \to \infty$,

$$
\frac{1}{T} \left( \sum_{0 < \gamma \leq T} h(\gamma \alpha) - N(T) \int h \right) = -\frac{1}{\pi} \Re \left\{ \sum_{\substack{|m\alpha| > 0 \\|m\| \leq J}} \frac{c_m \Lambda(x_m)}{\sqrt{x_m}} \right\} + o(1). \quad (3.5)
$$

Since $\Lambda(x_m) = 0$ unless $x_m$ is prime power. Similar to the proof of Theorem 1, by our definition of $g_\alpha$, (1.9) and (1.10), we see that, as $T \to \infty$, the right hand side of (3.5) is equal to

$$
\int_{T^n} h g_\alpha.
$$

Therefore, we prove the conclusion of the theorem.

**Proof of Theorem 5.**

For the special form of $\alpha$ when $r = n$, we can solve the above linear equations for $\alpha$, and find that each $2\pi \alpha_i$ is a linear combination of these $\log p_i$’s with rational coefficients. Hence, by Baker’s theorem, for such $\alpha$, there exist constants $D = D(\alpha)$ and $\mu$, such that, for all $m \in \mathbb{Z}^n$,

$$
|m \cdot \alpha| \geq \frac{D}{(\|m\| + 1)^\mu}. \quad (3.6)
$$

Thus, such $\alpha$ satisfies condition (1.11) for some constant $C$. The conclusion of the theorem follows from Theorem 4.

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