THE PRIME NUMBER RACE AND ZEROS OF \( L \)-FUNCTIONS OFF THE CRITICAL LINE

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Abstract. We examine the effects of certain hypothetical configurations of zeros of Dirichlet \( L \)-functions lying off the critical line on the distribution of primes in arithmetic progressions.

1. Introduction

Let \( \pi_{q,a}(x) \) denote the number of primes \( p \leq x \) with \( p \equiv a \pmod{q} \). The study of the relative magnitudes of the functions \( \pi_{q,a}(x) \) for a fixed \( q \) and varying \( a \) is known colloquially as the “prime race problem” or “Shanks-Rényi prime race problem”. Fix \( q \) and distinct residues \( a_1, \ldots, a_r \) with \((a_i, q) = 1\) for each \( i \). As colorfully described in the first paper of [KT1], consider a game with \( r \) players called “1” through “\( r \)”, and at time \( t \), each player “\( j \)” has a score of \( \pi_{q,a_j}(t) \) (i.e. player “\( j \)” scores 1 point whenever \( t \) reaches a prime \( \equiv a_j \pmod{q} \)). As \( t \to \infty \), will each player take the lead infinitely often? More generally, will all \( r ! \) orderings of the players occur for infinitely many integers \( t \)? It is generally believed that the answers to both questions is yes, for all \( q, a_1, \ldots, a_r \).

As first noted by Chebyshev [Ch] in 1853, some orderings may occur far less frequently than others (e.g. if \( q = 3, a_1 = 1, a_2 = 2 \), then player “1” takes the lead for the first time when \( t = 608,981,813,029 \) [BH]). More generally, when \( r = 2 \), \( a_1 \) is a quadratic residue modulo \( q \), and \( a_2 \) is a quadratic non-residue modulo \( q \), \( \pi_{q,a_2}(x) - \pi_{q,a_1}(x) \) tends to be positive more often than it is negative (this phenomenon is now called “Chebyshev’s bias”). In 1914, Littlewood [L] proved that both functions \( \pi_{4,3}(x) - \pi_{4,1}(x) \) and \( \pi_{3,2}(x) - \pi_{3,1}(x) \) change sign infinitely often. Later Knapowski and Turán ([KT1], [KT2]) proved for many \( q, a, b \) that \( \pi_{q,b}(x) - \pi_{q,a}(x) \) changes sign infinitely often. The distribution of the functions \( \pi_{q,a}(x) \) is closely linked with the distribution of the zeros in the critical strip \( 0 < \Re{s} < 1 \) of the Dirichlet \( L \)-functions \( L(s, \chi) \) for the characters \( \chi \) modulo \( q \). Some of the results of Knapowski and Turán are proved under the assumption that the

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functions $L(s, \chi)$ have no real zeros in $(0, 1)$, or that for some number $K_q$, the zeros of the functions $L(s, \chi)$ with $|3s| \leq K_q$ all have real part equal to $\frac{1}{2}$.

Theoretical results for $r > 2$ are more scant, all depending on the unproven Extended Riemann Hypothesis for $q$ (abbreviated ERH$_q$), which states that all these zeros lie on the critical line $\Re s = \frac{1}{2}$. Kaczorowski ([K1], [K2], [K3]) has shown that the truth of ERH$_q$ implies that for many $r$-tuples $(q, a_1, \cdots, a_r)$, $\pi_{q,a_1}(x) > \cdots > \pi_{q,a_r}(x)$ for arbitrarily large $x$. If, in addition to ERH$_q$, one assumes that the collection of non-trivial zeros of the $L$-functions for characters modulo $q$ are linearly independent over the rationals (GSH$_q$, the grand simplicity hypothesis), Rubinstein and Sarnak [RS] have shown that for any $r$-tuple of coprime residue classes $a_1, \ldots, a_r$ modulo $q$, that all $r!$ orderings of the functions $\pi_{q,a_1}(x)$ occur for infinitely many integers $x$. In fact they prove more, that the logarithmic density of the set of real $x$ for which any such inequality occurs exists and is positive.

In light of the results of Littlewood and of Knapowski and Turán, one may ask if such results for $r > 2$ may be proved without the assumption of ERH$_q$. In particular, can it be shown, for some quadruples $(q, a_1, a_2, a_3)$, that the 6 orderings of the functions $\pi_{q,a_1}(x)$ occur for infinitely many integers $x$, without the assumption of ERH$_q$ (while still allowing the assumption that zeros with imaginary part $< K_q$ lie on the critical line for some constant $K_q$)? In this paper we answer this question in the negative (in a sense) for all quadruples $(q, a_1, a_2, a_3)$. Thus, in a sense the hypothesis ERH$_q$ is a necessary condition for proving any such results when $r > 2$.

Let $C_q$ be the set of non-principal characters modulo $q$. Let $D = (q, a_1, a_2, a_3)$, where $a_1, a_2, a_3$ are distinct residues modulo $q$ which are coprime to $q$. Suppose for each $\chi \in C_q$, $B(\chi)$ is a sequence of complex numbers with positive imaginary part (possibly empty, duplicates allowed), and denote by $\mathcal{B}$ the system of $B(\chi)$ for $\chi \in C_q$. Let $n(\rho, \chi)$ be the number of occurrences of the number $\rho$ in $B(\chi)$. The system $\mathcal{B}$ is called a barrier for $D$ if the following hold:
(i) all numbers in each $B(\chi)$ have real part in $[\beta_2, \beta_3]$, where $\frac{1}{2} < \beta_2 < \beta_3 \leq 1$;
(ii) for some $\beta_1$ satisfying $\frac{1}{2} \leq \beta_1 < \beta_2$, if we assume that for each $\chi \in C_q$ and $\rho \in B(\chi)$, $L(s, \chi)$ has a zero of multiplicity $n(\rho, \chi)$ at $s = \rho$, and all other zeros of $L(s, \chi)$ in the upper half plane have real part $\leq \beta_1$, then one of the six orderings of the three functions $\pi_{q,a_1}(x)$ does not occur for large $x$.

If each sequence $B(\chi)$ is finite, we call $\mathcal{B}$ a finite barrier for $D$ and denote by $|\mathcal{B}|$ the sum of the number of elements of each sequence $B(\chi)$, counted according to multiplicity.

**Theorem 1.** For every real numbers $\tau > 0$ and $\sigma > \frac{1}{2}$ and every $D = (q, a_1, a_2, a_3)$, there is a finite barrier for $D$, where each sequence $B(\chi)$ consists of numbers with real part $\leq \sigma$ and imaginary part $> \tau$. In fact, for most $D$, there is a barrier with $|\mathcal{B}| \leq 3$.

We do not claim that the falsity of ERH$_q$ implies that one of the six orderings does not occur for large $x$. For example, take $\sigma > \frac{1}{2}$, and suppose each non-principal character modulo $q$ has a unique zero with positive imaginary part to the right of the critical line, at $\sigma + i\gamma_\chi$. If the numbers $\gamma_\chi$ are linearly independent over the rationals, it follows easily from Lemma 1.1 below and the Kronecker-Weyl Theorem.
that in fact all \( \phi(q) \) orderings of the functions \( \{ \pi_{q,a}(x) : (a, q) = 1 \} \) occur for an unbounded set of \( x \).

We now present a general formula for \( \pi_{q,a}(x) \) in terms of the zeros of the functions \( L(s, \chi) \). Throughout this paper, constants implied by the Landau \( O \)- and Vinogradov \( \ll \)- symbols may depend on \( q \), but not on any other variable.

**Lemma 1.1.** Let \( \beta \geq \frac{1}{2} \), \( x \geq 10 \) and for each \( \chi \in \mathbb{C}_q \), let \( B(\chi) \) be the sequence of zeros (duplicates allowed) of \( L(s, \chi) \) with \( \Re s > \beta \) and \( \Im s > 0 \). Suppose further that all \( L(s, \chi) \) are zero-free on the real segment \( 0 < s < 1 \). If \( (a, q) = (b, q) = 1 \) and \( x \) is sufficiently large, then

\[
\phi(q) \left( \pi_{q,a}(x) - \pi_{q,b}(x) \right) = -2\Re \left[ \sum_{\chi \in \mathbb{C}_q} (\overline{\chi}(a) - \overline{\chi}(b)) \sum_{\rho \in B(\chi), |\Re \rho| \leq x} f(\rho) \right] + O(x^{\beta} \log^2 x),
\]

where

\[
f(\rho) := \frac{x^\rho}{\rho \log x} + \frac{1}{\rho^2} \int_2^x \frac{t^\rho}{t \log^2 t} \, dt = \frac{x^\rho}{\rho \log x} + O \left( \frac{x^{\Re \rho}}{|\rho|^2 \log^2 x} \right).
\]

**Proof.** Let \( \Lambda(n) \) be the von Mangoldt function, and define

\[
\Psi_{q,a}(x) = \sum_{n \leq x \atop n \equiv a \pmod{q}} \Lambda(n), \quad \Psi(x; \chi) = \sum_{n \leq x} \Lambda(n) \chi(n).
\]

Let \( D_q \) be the set of all Dirichlet characters \( \chi \) modulo \( q \). Then

\[
\pi_{q,a}(x) = \sum_{n \leq x \atop n \equiv a \pmod{q}} \frac{\Lambda(n)}{\log n} + O(x^{1/2})
\]

\[
= \int_{2-}^x d\Psi_{q,a}(t) \quad \log t + O(x^{1/2})
\]

\[
= \frac{\Psi_{q,a}(x)}{\log x} + \int_2^x \frac{\Psi_{q,a}(t)}{t \log^2 t} \, dt + O(x^{1/2})
\]

\[
= \frac{1}{\phi(q)} \sum_{\chi \in D_q} \overline{\chi}(a) \left( \frac{\Psi(x; \chi)}{\log x} + \int_2^x \frac{\Psi(t; \chi)}{t \log^2 t} \, dt \right) + O(x^{1/2}).
\]

Then

\[
(1.1) \quad \phi(q)(\pi_{q,a}(x) - \pi_{q,b}(x)) = \sum_{\chi \in \mathbb{C}_q} (\overline{\chi}(a) - \overline{\chi}(b)) \left( \frac{\Psi(x; \chi)}{\log x} + \int_2^x \frac{\Psi(t; \chi)}{t \log^2 t} \, dt \right) + O(x^{1/2}).
\]
By well-known explicit formulas (Ch. 19, (7)-(8) in [D]), when $\chi \in C_q$,

\[
\Psi(x; \chi) = -\sum_{|\Re \rho| \leq x} \frac{x^\rho}{\rho} + O \left( \log^2 x \right),
\]

where the sum is over zeros $\rho$ of $L(s, \chi)$ with $0 < \Re \rho < 1$. Since the number of zeros with $0 \leq \Re \rho \leq T$ is $O(T \log T)$ ([D], Ch. 16, (1)), by partial summation we have

\[
\sum_{0 < \Re \rho \leq x \atop \Re \rho \leq \beta} \left| \frac{x^\rho}{\rho} \right| \leq x^\beta \sum_{0 < \Re \rho \leq x} \frac{1}{|\rho|} \leq x^\beta \log^2 x.
\]

The implied constant depends on the character, and hence only on $q$. By (1.2),

\[
\Psi(x; \chi) = -\sum_{|\Re \rho| \leq x \atop \Re \rho > \beta} \frac{x^\rho}{\rho} + O \left( x^\beta \log^2 x \right),
\]

The first part of the lemma follows by inserting (1.3) into (1.1) and combining zeros $\rho$ of $L(s, \chi)$ and $\overline{\chi}$ of $L(s, \overline{\chi})$. Lastly, if $\frac{1}{2} \leq \sigma = \Re \rho$, integration by parts gives

\[
\int_2^x \frac{t^\rho}{t \log^2 t} \, dt = \frac{t^\rho}{\rho \log^2 t} \bigg|_2^x + 2 \int_2^x \frac{t^{\rho-1}}{\log^3 t} \, dt
\]

\[
\leq \frac{x^\sigma}{|\rho| \log^2 x} + \frac{1}{|\rho|} \left[ \frac{1}{\log^3 2} \int_2^{\sqrt{x}} t^\sigma \, dt + \frac{8}{\log^3 x} \int_x^{\sqrt{x}} t^{\sigma-1} \, dt \right]
\]

\[
\leq \frac{x^\sigma}{|\rho| \log^2 x}.
\]

This completes the proof of the lemma. $\square$

In the next three sections, we show several methods for constructing barriers, which, by Lemma 1.1, boils down to analyzing the two functions

\[
\Re \sum_{\chi \in C_q} (\chi(a_j) - \overline{\chi}(a_j)) \sum_{\rho \in \mathcal{B}(\chi)} \frac{x^\rho}{\rho} \quad (j = 1, 2).
\]

In section 2 we construct a barrier using two simple zeros (one of which may be a zero for several characters). Section 3 details a method using a zero for $L(s, \chi)$ and a zero for $L(s, \chi^2)$ (for most $D$ these are simple or double zeros). Lastly, section 4 presents a more general method with two numbers, which are zeros for each character of certain high multiplicities. Together, the three constructions provide barriers for all quadruples $(a_1, a_2, a_3, a_4)$.

All of the constructions in sections 2–4 involve two zeros, one with imaginary part $t$ and the other with imaginary part $2t$. Thus, we assume that both ERH$_q$ and GSH$_q$ are false. Answering a question posed by Peter Sarnak, in section 5 we
construct a barrier (with an infinite set \( B(\chi) \)) where the imaginary parts of the numbers in the sets \( B(\chi) \) are linearly independent; in particular, we assume all zeros of each \( L(s, \chi) \) are simple, and \( L(s, \chi_1) = 0 = L(s, \chi_2) \) does not occur for \( \chi_1 \neq \chi_2 \) and \( \Re s > \beta_2 \).

We adopt the notations \( e(z) = e^{2\pi i z} \), \([x]\) is the greatest integer \( \leq x \), \([x]\) is the least integer \( \geq x \), \( \{x\} = x - [x] \) is the fractional part of \( x \), and \( \|x\| \) is the distance from \( x \) to the nearest integer. Also, \( \arg z \) is the argument of the nonzero complex number \( z \) lying in \( [-\pi, \pi) \). Throughout, \( q = 5 \) or \( q \geq 7 \), and \( (a_1, q) = (a_2, q) = (a_3, q) = 1 \).

2. First Construction

**Lemma 2.1.** If, for some relabelling of the numbers \( a_i \), there is a set \( S \) of non-principal Dirichlet characters modulo \( q \) such that

\[
\sum_{\chi \in S} \chi(a_1) = \sum_{\chi \in S} \chi(a_2) \neq \sum_{\chi \in S} \chi(a_3),
\]

then there is a barrier \( \mathcal{B} \) for \( D = (q, a_1, a_2, a_3) \) with \( |\mathcal{B}| \leq |S| + 1 \).

**Remark.** The hypotheses of Lemma 2.1 are satisfied when, for example, \( q \) has a primitive root \( g \), and \( a_3/a_2 \) is not in the subgroup of \( (\mathbb{Z}/q\mathbb{Z})^* \) generated by \( a_2/a_1 \). Writing \( a_2/a_1 \equiv g^f \), we take the character with \( \chi(g) = e(1/(f, \phi(q))) \) and \( S = \{\chi\} \).

**Proof.** Suppose \( 1/2 \leq \beta < \sigma_2 < \sigma_1 \leq \min(\sigma, 0.501) \), and let \( \chi_2 \) be a character with \( \chi_2(a_1) \neq \chi_2(a_2) \) (\( \chi_2 \) may or may not be in \( S \)). Let \( T_q \) be a large number, depending only on \( q \). Let \( \rho_1 = \sigma_1 + it \), \( \rho_2 = \sigma_2 + 2it \) where \( t > T_q \). Suppose \( L(s, \chi) \) has a simple zero at \( s = \rho_1 \) for each \( \chi \in S \), \( L(s, \chi_2) \) has a simple zero at \( s = \rho_2 \), and no other non-trivial zeros of any \( L \)-function in \( C_q \) have real part exceeding \( \beta \). Let

\[
D_1(x) := \phi(q)(\pi_{q,a_1}(x) - \pi_{q,a_2}(x)), \quad D_2(x) := \phi(q)(\pi_{q,a_3}(x) - \pi_{q,a_2}(x)).
\]

By Lemma 1.1 and our hypotheses, if \( x \) is sufficiently large,

\[
D_1(x) = \frac{2x^{\sigma_2}}{\log x} \left[ \Re \left( \frac{e^{2it \log x}}{\sigma_2 + 2it} W \right) + O \left( \frac{1}{\log x} \right) \right], \quad W = \overline{\chi_2(a_2)} - \overline{\chi_2(a_1)},
\]

\[
D_2(x) = \frac{2x^{\sigma_1}}{\log x} \left[ \Re \left( \frac{e^{it \log x}}{\sigma_1 + it} Z \right) + O \left( \frac{1}{\log x} \right) \right], \quad Z = \sum_{\chi \in S} (\overline{\chi(a_2)} - \overline{\chi(a_3)}).
\]

Define

\[
A(x) = \left\| \frac{1}{\pi} \arg \left( \frac{e^{it \log x}}{\sigma_1 + it} Z \right) - \frac{1}{2} \right\| = \left\| \frac{1}{\pi} (t \log x + \arg Z + \tan^{-1}(\sigma_1/t)) \right\|.
\]
If \( A(x) \geq (\log x)^{-1/2} \), then \(|D_2(x)| \gg x^{\alpha_1}/\log^{3/2} x\). But \( D_1(x) = O(x^{\alpha_2})\), so for such \( x \), \( \pi_{q,a_3}(x) \) is either the largest or the smallest of the three functions. When \( A(x) < (\log x)^{-1/2} \), then

\[
C(x) := \arg \left( \frac{e^{2it\log x}}{\sigma_2 + 2it} W \right) \\
\equiv \arg W - \frac{\pi}{2} + \tan^{-1} \left( \frac{\sigma_2}{2t} \right) + 2t \log x \\
\equiv \arg W + \tan^{-1} \left( \frac{\sigma_2}{2t} \right) - 2 \arg Z - 2 \tan^{-1} \left( \frac{\sigma_1}{t} \right) + O \left( \frac{1}{\sqrt{\log x}} \right) \\
\equiv \arg W - 2 \arg Z - F(x) \pmod{\pi},
\]

where \( 1/(2t) < F(x) < 1/t \) for large \( x \). The number of possibilities for \( \arg W - 2 \arg Z \) depends only on \( q \), hence we may assume either

\[
B = \left\{ \frac{1}{\pi}(\arg W - 2 \arg Z) \right\} - \frac{1}{2}
\]
satisfies either \( B = 0 \) or \(|B| > 2/t \geq 2F(x)\) (by taking \( T_q \) sufficiently large). We have

\[
C(x) \equiv \pi B + \frac{\pi}{2} - F(x) \pmod{\pi}.
\]

If \( B = 0 \), then \( C(x) \) is either \( \pi/2 - F(x) \) or \( 3\pi/2 - F(x) \pmod{2\pi} \), whence \( D_1(x) \) takes only one sign for such \( x \). Likewise, \( C(x) \in (\pi/2 + 2/t, \pi) \) if \( B > 2/t \) and \( C(x) \in (-F(x), \pi/2 - 2/t) \) if \( B < -2/t \). In all cases, when \( A(x) < (\log x)^{-1/2} \), \( D_1(x) \) takes only one sign. Therefore, one of the orderings \( \pi_{q,a_1}(x) > \pi_{q,a_3}(x) > \pi_{q,a_2}(x) \) or \( \pi_{q,a_3}(x) > \pi_{q,a_2}(x) > \pi_{q,a_1}(x) \) does not occur for large \( x \).

**Remark.** By similar reasoning, for any integer \( k \geq 2 \) one may construct a barrier with one zero having imaginary part \( t \) and another zero having imaginary part \( kt \).

### 3. Second Construction

The basic idea of this section is to find a character \( \chi \) so that the values \( \chi(a_1), \chi(a_2), \chi(a_3) \) are nicely spaced around the unit circle, but not too well spaced (e.g. cube roots of 1 or translates thereof). In almost all circumstances we can find such a character.

**Lemma 3.1.** Let \( s_1 = \text{ord}_q(a_2/a_1), s_2 = \text{ord}_q(a_3/a_2) \) and \( s_3 = \text{ord}_q(a_1/a_3) \). If one of \( s_1, s_2, s_3 \) is not in \( \{3, 7, 13, 21\} \), then for some relabeling of the \( a_i \)'s, there is a Dirichlet character \( \chi \) satisfying either

(i) \( \chi(a_1) = \chi(a_2) \neq \chi(a_3) \); or

(ii) \( \chi(a_i) = e(r_i) \) with \( 0 \leq r_1 < r_2 < r_3 < 1 \), and \( d_1 = r_2 - r_1, d_2 = r_3 - r_2 \) satisfy

\[
1/3 < d_1 \leq d_2 < 1/2, \quad \text{or} \quad (d_1, d_2) \in \left\{ \left( \frac{6}{19}, \frac{9}{19} \right), \left( \frac{12}{37}, \frac{16}{37} \right) \right\}.
\]
Remark. In the case that (i) holds, the hypotheses of Lemma 2.1 hold with $S = \{\chi\}$, and thus there is a finite barrier for $D$ with $|\mathcal{B}| = 2$. Therefore, in this section we confine ourselves with the case that (ii) holds (Lemma 3.5 below).

Before proving Lemma 3.1, we begin with some simple lemmas about the existence of characters with certain properties.

Lemma 3.2. Suppose $q \geq 3$ and $(b, q) = 1$. Let $m$ be the order of $b$ modulo $q$. Then there is a Dirichlet character $\chi$ modulo $q$ with $\chi(b) = e(1/m)$.

Proof. Suppose $g_1, \ldots, g_t$ generate $\mathbb{Z}/g\mathbb{Z}$ and $b = g_1^{f_1} \cdots g_t^{f_t}$. Let $s_i = \text{ord}_q g_i$ for each $i$, and $s'_i$ be the order of $g_i^{f_i}$. Then $s_i' = s_i/(f_i, s_i)$ and $m = \text{lcm}[s'_1, \ldots, s'_t]$. Let $f'_i = f_i/(f_i, s_i)$, so in particular $(s'_i, f'_i) = 1$. The gcd of the $t + 1$ numbers $m, f'_i/m, s'_i$ is 1, so there are integers $h_1, \ldots, h_t$ so that $\sum h_i f'_i/m s'_i \equiv 1 \pmod{m}$. Take the character $\chi$ with $\chi(g_i) = e(h_i/s_i)$ for each $i$, then $\chi(b) = \prod \chi(g_i)^{f_i} = e(h_1 f'_1/s'_1 + \cdots + h_t f'_t/s'_t) = e(1/m)$. \hfill \Box

Lemma 3.3. Suppose $b, c$ are distinct residues modulo $q$ with $(b, q) = (c, q) = 1$. Suppose that $r \mid \text{ord}_q b$ and for every $p^n \mid r$ with $a \geq 1$, $p^{a+1} \nmid \text{ord}_q c$. Then there is a Dirichlet character $\chi$ modulo $q$ such that

$$\chi(b) = e(1/r), \quad \chi(c)^r = 1.$$  

Proof. Let $s_1 = \text{ord}_q b$ and $s_2 = \text{ord}_q c$. By Lemma 3.2, there is a character $\chi_1$ with $\chi_1(b) = e(1/s_1)$ and therefore a character $\chi_2$ with $\chi_2(b) = e(1/r)$. Since $c$ has order $s_2$, $\chi_2(c) = e(g/s_2)$ for some integer $g$. Write $s_2 = uv$ where $(u, r) = 1$ and $v \mid r$. Define $x$ by $xu \equiv 1 \pmod{r}$, and let $\chi = \chi_2^x$. Then $\chi(b) = \chi_2(b)^{xu} = e(1/r)$ and $\chi(c) = e(gxu/s_2) = e(gx/v) = e(gx(r/v)/r)$. \hfill \Box

Definition. An odd number $m$ is “good” if for every $j$, $1 \leq j \leq m - 1$, there is a number $k$ such that among the points $(0, k/m, kj/m) \pmod{1}$, either two are equal (and not equal to the third), or two of the three distances $d_1, d_2, d_3$ (with sum = 1) between the points satisfy (3.1).

Remark. To prove that a number $m$ is good, we need only to check $2 \leq j \leq (m + 1)/2$, since for $j = 1$ we take $k = 1$, and if $k$ works for $j = j_0$ then the same $k$ works for $j = m + 1 - j_0$.

Lemma 3.4. Every odd prime $p$ except $p \in P = \{3, 7, 13\}$ is good, and for $p \in P$, $p^2$ is good. Also, the numbers 39, 91 and 273 are good.

Proof. A short computation implies that if $p \in P$, then $p$ is not good, but $p^2$ is good. Also, by a short computation, all other odd primes $\leq 83$ are good, as well as 39, 91 and 273. The following $j$ values have no associated $k$-value: for $m = 3$, $j = 2$; for $m = 7, j = 3, 5$; for $m = 13; j = 3, 5, 6, 8, 9, 11$; for $m = 21, j = 5, 17$.

Suppose that $m = p > 84$ is prime and write each product $kj = \ell p + r$ with $0 \leq r < p$. We shall prove that for each $j \in [2, \frac{p^2+1}{2}]$, there is a $k$ so that two of the three distances satisfy $\frac{1}{3} < d_1 < d_2 < \frac{1}{2}$. We now divide up the $j \in [2, \frac{p^2+1}{2}]$ into 9 cases:
Case I. $j \in \{3, 5, 7, \frac{p+1}{2}\}$. For $j = 3$ take $p/6 < k < 2p/9$ and for $j = 5, 7$ take any $k$ with $p/(2j) < k < p/(2j - 2)$. There is such a $k$ when $p > 84$. Then $p - jk$ and $jk - k$ both lie in $(p/3, p/2)$. For $j = \frac{p+1}{2}$ take $k = 2 \lfloor p/3 \rfloor$, then $r = \lfloor p/3 \rfloor$, so both $r$ and $k - r$ lie in $(p/3, p/2)$ for $p > 6$.

Case II. $9 < j < p/6 + 1$. Take $m = \lfloor \frac{5}{12}(j - 1) \rfloor$. Then

\[
\frac{m + 1/2}{j - 1} < \frac{5}{12} + \frac{1/2}{j - 1} < \frac{1}{2}, \quad \frac{m + 1/3}{j - 1} \geq \frac{5}{12} - \frac{7/12}{j - 1} > 1/3.
\]

Therefore, if

\[
\frac{p(m + 1/3)}{j - 1} < k < \frac{p(m + 1/2)}{j - 1},
\]

then $k$ and $r - k$ lie in $(p/3, p/2)$. But the above interval has length $p/(6j - 6) > 1$, so such a $k$ exists.

Case III. $2 < j < p/3 + 1, j$ even. Take $k = \frac{p-1}{2}$. Then $r = p - j/2$ and both $k$ and $r - k$ lie in $(p/3, p/2)$.

Case IV. $p/3 + 1 < j < 3p/7, j$ even. Take $h$ so that $1 \leq h < \frac{p-3}{18}$ and

\[
\frac{2h + 2/3}{6h + 1} < j < \frac{2h + 1}{6h + 1}p.
\]

The largest admissible $h$ is at least $\frac{p-19}{18}$, so the above intervals cover $(\frac{p(p-13)}{3(p-16)} \cdot 3p/7)$, which contains $[\frac{p+4}{3}, 3p/7]$ for $p > 64$. Then take $k = \frac{p-1}{2} - 3h$, so that $r \in (p/2, 2p/3)$.

Case V. $2p/5 + 1 < j \leq \frac{p-1}{2}, j$ even. We take $h$ so that $1 \leq h < \frac{p-3}{12}$ and

\[
\frac{2h}{4h + 1} < j - 1 < \frac{2h + 1/3}{4h + 1}p.
\]

The largest admissible $h$ is at least $\frac{p-13}{12}$, so these intervals cover $(2p/5, \frac{p(p-11)}{2(p-10)})$, which includes $(2p/5, \frac{p-3}{2})$ for $p > 13$. Then take $k = \frac{p-1}{2} - 2h$, so that $r - k \in (p/3, p/2)$.

Case VI. $p/3 + 1 < j < \frac{p-1}{2}, j$ odd. Take $h$, $0 \leq h < \frac{p-15}{12}$ so that

\[
\frac{2h + 1}{4h + 3} < j - 1 < \frac{2h + 4/3}{4h + 3}p.
\]

Then take $k = \frac{p-3}{2} - 2h$, so that $k - r \in (p/3, p/2)$. The above intervals cover $(p/3, \frac{p-3}{2})$ provided that $p > 24$.

Case VII. $p/3 - 1 < j < p/3 + 1$. Write $j = \frac{p+t}{3}$, where $-2 \leq t \leq 2$, $t \neq 0$. Here we take $k = 3 \lfloor p/9 \rfloor + b$, where $0 \leq b < 2$ and $t + 3b \equiv w \pmod{9}$, $w \in \{5, 7\}$. If $p > 28$ then $k \in (p/3, p/2)$. If $w = 5$, then $r = 5p/9 + E$, where
\[ |E| \leq 22/9. \] Thus, \( r \in (p/2, 2p/3) \) when \( p > 44 \). When \( w = 7, t = 1, b = 2 \), then \( r \in (7p/9, 7p/9 + 14/9) \).

**Case VIII.** \( 5p/21 < j < p/3 - 1, j \) odd. Take \( 1 \leq h < \frac{p-3}{18} \) so that

\[
\frac{6h - 1}{18h + 3} p < j < \frac{2h}{6h + 1} p.
\]

Take \( k = \frac{p-1}{2} - 3h \), so \( r \in (p/3, p/2) \). The above intervals cover \( (5p/21, p/3 - 1) \).

**Case IX.** \( p/6 + 1 < j < 5p/21, j \) odd. If \( p/5 < j - 1 < 4p/15 \), take \( k = \frac{p-5}{2} \), so that \( r \in (5p/6 - 5/2, p - 5/2) \). If \( p/7 < j - 1 < 4p/21 \), then \( k = \frac{p-7}{2} \) works and if \( 5p/27 < j < 2p/9 \) then \( k = \frac{p-9}{2} \) works. \( \square \)

**Proof of Lemma 3.1.** By hypothesis, there are two possibilities:

(i) some \( s_i \) (say \( s_1 \)) is divisible by a prime power \( p^u \) other than \( 3, 7, \) or \( 13 \);

(ii) Each \( s_i \) divides \( 273 \) and some \( s_i \) (say \( s_1 \)) equals \( 39, 91 \) or \( 273 \).

Say \( s_1 \) is divisible by \( p \), with \( p^w+1 \nmid s_2 \) and \( p^w+1 \nmid s_3 \). By Lemma 3.3, there is a character \( \chi_1 \) with \( \chi_1(a_2/a_1) = e(1/p^w) \) and \( \chi_1(a_3/a_2) = e(m/p^w) \) for some integer \( m \). If \( p = 2 \), let \( \chi = \chi_1^{w+1} \), so that \( \chi(a_2/a_1) = -1 \) and

\[ 1 = \chi(a_2/a_1)\chi(a_3/a_2)\chi(a_1/a_3) = -\chi(a_3/a_2)\chi(a_1/a_3). \]

But each character value on the right is either -1 or 1, so either \( \chi(a_2) = \chi(a_3) \) or \( \chi(a_1) = \chi(a_3) \) and (i) is satisfied. If \( p \) is odd, let \( \chi_2 = \chi_1^{p^w-1} \) if \( p \notin P \) and \( \chi_2 = \chi_1^{p^{u-2}} \) if \( p \in P \). Then \( \chi_2(a_2/a_1) = e(1/p^u) \), where \( u = 2 \) if \( p \in P \) and \( u = 1 \) otherwise. Write \( \chi_2(a_3/a_2) = e(j/p^u) \). If \( j = 0 \) then \( \chi_2(a_2) = \chi_2(a_3) \) and (i) is satisfied. Otherwise, since \( p^u \) is good by Lemma 3.4, there is a number \( k \) so that two of the three distances of the points \((0, k/p^u, kj/p^u) \) (mod 1) satisfy (3.1). Taking \( \chi = \chi_2^k \) gives (ii) for some relabeling of the \( a_i \)'s.

In the case that each \( s_i \) divides \( 273 \) and \( s_1 \in \{39, 91, 273 \} \), by Lemma 3.3 there is a character \( \chi_1 \) with \( \chi_1(a_2/a_1) = e(1/r) \) and \( \chi_1(a_3/a_2) = e(g/r) \) for some integer \( g \). (here \( r = s_1 \)). Since \( r \) is good by Lemma 3.4, there is a \( k \) such that two of the three distances of the points \((0, k/r, kj/r) \) (mod 1) satisfy (3.1). Taking \( \chi = \chi_1^k \) gives (ii) for some relabeling of the \( a_i \)'s. \( \square \)

**Lemma 3.5.** Suppose that for some relabeling of \( a_1, a_2, a_3 \) and some Dirichlet character \( \chi \) modulo \( q \), \( \chi(a_i) = e(r_i) \) with \( 0 \leq r_1 < r_2 < r_3 \leq 2, d_1 = r_2 - r_1 \) and \( d_2 = r_3 - r_2 \) and \((d_1, d_2) \) satisfies (3.1). Then there is a finite barrier \( B \) for \( D = (q, a_1, a_2, a_3) \) with \( |B| \leq 14 \). If \( d_1 > \frac{1}{5} \), then \( |B| \leq 3 \).

**Proof.** For some \( 1/2 < \beta < \alpha \leq \sigma \) and large \( \gamma \), suppose \( L(s, \chi) \) has a zero at \( s = \alpha + i\gamma \) of order \( c_1 \), and \( L(s, \chi^2) \) has a zero at \( s = \alpha + 2i\gamma \) of order \( c_2 \), where

\[
(c_1, c_2) = \left\{ \begin{array}{ll}
(1, 2) & d_1 > \frac{1}{3} \\
(5, 9) & d_1 = \frac{6}{19} \\
(3, 5) & d_1 = \frac{12}{37}.
\end{array} \right.
\]
Suppose all other non-trivial zeros of $L$-functions modulo $q$ have real part $\leq \beta$. Let

$$D_1(x) = \frac{\phi(q) \log x}{x^\alpha} (\pi_{q,a_2}(x) - \pi_{q,a_1}(x)),$$

$$D_2(x) = \frac{\phi(q) \log x}{x^\alpha} (\pi_{q,a_3}(x) - \pi_{q,a_2}(x)).$$

Let $u = \log x$. For large $x$, Lemma 1.1 and the identity

$$\sin(a - b) - \sin(a - c) = 2 \cos(a - \frac{b + c}{2}) \sin(\frac{b - c}{2})$$

give

$$D_1(x) = \frac{4}{\gamma} \sum_{\ell=1}^{2} \frac{c_\ell}{\ell} \sin(d_1 \ell \pi) \cos(\ell \gamma u - (r_1 + r_2) \pi \ell) + O(1/\gamma^2),$$

(3.2)

$$D_2(x) = \frac{4}{\gamma} \sum_{\ell=1}^{2} \frac{c_\ell}{\ell} \sin(d_2 \ell \pi) \cos(\ell \gamma u - (r_2 + r_3) \pi \ell) + O(1/\gamma^2).$$

For $j = 1, 2$ define

$$g_j(y) = c_1 \sin(\pi d_j) \cos y + \frac{c_2}{2} \sin(2 \pi d_j) \cos 2y$$

$$= c_1 \sin(\pi d_j) \left( \cos y + \frac{\varepsilon_j}{c_1} \cos(\pi d_j) \cos 2y \right).$$

Because $0 < d_j < 1/2$, $\cos \pi d_j$ and $\sin \pi d_j$ are both positive. We claim that

$$\min(g_1(\gamma u - (r_1 + r_2) \pi), g_2(\gamma u - (r_2 + r_3) \pi)) < 0 \quad (u \geq 0),$$

which is equivalent to showing

$$\min(g_1(y), g_2(y - \pi(d_1 + d_2))) < 0$$

for all real $y$. Since $g_1$ and $g_2$ are periodic and continuous, in fact the minimum above is $\leq -\delta$ for some $\delta > 0$. If $\gamma$ is large (depending on $\delta$), this implies that one of the two functions on the left in (3.2) is negative for all large $x$. Thus for large $x$, $\pi_{q,a_3}(x) > \pi_{q,a_2}(x) > \pi_{q,a_1}(x)$ does not occur.

To prove (3.3), we consider the one parameter family of functions $h(y; \lambda) = \cos y + \lambda \cos(2y)$ for $0 < \lambda < 1$. These are all even functions, so it suffices to look at $0 \leq y \leq \pi$. We have $h(y; \lambda)$ positive for $0 \leq y < v_\lambda$ and negative for $v_\lambda < y \leq \pi$, where $v_\lambda = \cos^{-1}[-1 + \sqrt{8 \lambda^2 + 1}]$. As a function of $\lambda$, $v_\lambda$ decreases from $\pi/2$ at $\lambda = 0$ to $\pi/3$ at $\lambda = 1$. For $i = 1, 2$, let $z_i = v_{\lambda_i}$ for $\lambda_i = (c_2/c_1) \cos \pi d_i$. Since $\pi(d_1 + d_2) < \pi$, (3.3) will follow from

$$z_1 + z_2 < \pi(d_1 + d_2).$$

(3.4)

When $(d_1, d_2) \in \{(\frac{6}{19}, \frac{9}{19}), (\frac{12}{37}, \frac{16}{37})\}$, (3.4) follows by direct calculation. When $\frac{1}{3} < d_1$, we have $c_1 = 1$, $c_2 = 2$ and $\lambda_j = 2 \cos \pi d_j$ ($j = 1, 2$). We claim for $j = 1, 2$ that $z_j < \pi d_j$, or equivalently $\cos z_j > \cos \pi d_j = \frac{1}{2} \lambda_j$. Since $0 < \lambda_j < 1$,

$$\cos z_j = \sqrt{\frac{8 \lambda_j^2 + 1 - 1}{4 \lambda_j}} > \sqrt{\frac{4 \lambda_j^4 + 4 \lambda_j^2 + 1 - 1}{4 \lambda_j}} = \frac{\lambda_j}{2},$$

which proves (3.4) in this case as well. \(\Box\)

Combining Lemmas 3.1 and 3.5 gives the following.
Corollary 3.6. Let \( s_1 = \text{ord}_q(a_2/a_1) \), \( s_2 = \text{ord}_q(a_3/a_2) \) and \( s_3 = \text{ord}_q(a_1/a_3) \). If one of \( s_1, s_2, s_3 \) is not in \( \{3, 7, 13, 21\} \), then there is a finite barrier \( B \) for \( D \) with \( |B| \leq 14 \).

4. Third Construction

Throughout this section, we assume that \( a_1, a_2, a_3 \) do not satisfy the conditions of Lemma 2.1.

Lemma 4.1. Let \( \chi \) be a character modulo \( q \) such that there are at least two different values among \( \chi(a_1), \chi(a_2), \chi(a_3) \). Then the following hold:
(a) \( \chi(a_1), \chi(a_2), \chi(a_3) \) are distinct;
(b) \( \Re \chi(a_1), \Re \chi(a_2), \Re \chi(a_3) \) are distinct;
(c) All the values \( \chi(a_1), \chi(a_2), \chi(a_3) \) are not \( \pm 1 \).
(d) \( \chi \) has order \( \geq 7 \).

Proof. (a) If this does not hold, the conditions of Lemma 2.1 hold with \( S = \{\chi\} \).
(b) If \( \chi(a_1) = \overline{\chi}(a_2) \), then, by (a), \( \Re \chi(a_3) \neq \Re \chi(a_1) \), and the conditions of Lemma 2.1 hold for \( S = \{\chi, \overline{\chi}\} \).
(c) If \( \chi(a_3) = 1 \) and \( k \) is the order of the character \( \chi \), then the conditions of Lemma 2.1 hold for \( S = \{\chi, \chi^2, \ldots, \chi^{k-1}\} \). If \( \chi(a_3) = -1 \) and none of \( \chi(a_i) = 1 \), then \( \chi^2(a_3) = 1 \neq \chi^2(a_1) \), and the conditions of Lemma 2.1 hold for \( S = \{\chi^2, \chi^4, \ldots, \chi^{2h-2}\} \) where \( h \) is the order of \( \chi^2 \).
(d) This follows directly from (b) and (c).

Lemma 4.2. There exists a character \( \chi \) modulo \( q \) of order \( \geq 7 \) such that

\[
\Re(\chi(a_3) - \chi(a_2))\Re(\chi^2(a_2) - \chi^2(a_1)) \neq \Re(\chi(a_2) - \chi(a_1))\Re(\chi^2(a_3) - \chi^2(a_2))
\]

and for some integers \( h, k \) with \( 1 \leq h < k \leq 3 \),

\[
\Im(\chi^h(a_3) - \chi^h(a_2))\Im(\chi^k(a_2) - \chi^k(a_1)) \neq \Im(\chi^h(a_2) - \chi^h(a_1))\Im(\chi^k(a_3) - \chi^k(a_2)).
\]

Proof. Let \( \chi \) be any character modulo \( q \) such that \( \chi(a_2/a_1) \neq 1 \). By Lemma 4.1 (a), the values \( \chi(a_1), \chi(a_2), \chi(a_3) \) are distinct. Denote \( \chi(a_j) = e^{2\pi i \varphi_j} \) (\( j = 1, 2, 3 \)). By Lemma 4.1 (b), the values \( \cos(\varphi_1), \cos(\varphi_2), \cos(\varphi_3) \) are distinct. Therefore, the matrix \( A = \cos^t(\varphi_j)_{j=1,2,3} \) is nonsingular. Since \( \cos(2\varphi) = 2\cos^2(\varphi) - 1 \), the matrix \( \cos(\ell \varphi_j)_{j=1,2,3} \) is also nonsingular, and this implies (4.1).

Next, by Lemma 4.1 (c), \( \sin(\varphi_j) \neq 0 \) (\( j = 1, 2, 3 \)). Therefore, the matrix \( B = \sin(\varphi_j) \cos^t(\varphi_j)_{j=1,2,3} \) is nonsingular. Using the identities \( \sin(2\varphi) = 2\sin(\varphi)\cos(\varphi) \), \( \sin(3\varphi) = 2\sin(\varphi)(4\cos^2(\varphi) - 1) \), it follows that the matrix \( \sin(\ell \varphi_j)_{j=1,2,3} \) is also nonsingular. This implies (4.2).
Lemma 4.3. Let $z_1$ and $z_2$ be complex numbers. We can associate with each $\chi \in C_q$ a non-negative real number $\lambda_\chi$ such that

$$z_1 = \sum_{\chi \in C_q} \lambda_\chi (\overline{\chi}(a_2) - \overline{\chi}(a_1)),$$

$$z_2 = \sum_{\chi \in C_q} \lambda_\chi (\overline{\chi}(a_3) - \overline{\chi}(a_2)).$$

(4.3)

Proof. Write $z_j = u_j + iv_j$ $(j = 1, 2)$, where $u_1, u_2, v_1, v_2$ are real. By Lemma 4.2, there is a character $\chi = \chi_0$ for which (4.1) and (4.2) hold. Thus, we can find real numbers $\lambda_1$ and $\lambda_2$ such that

$$\lambda_1 \Re(\chi_0(a_2) - \chi_0(a_1)) + \lambda_2 \Re(\chi_0^2(a_2) - \chi_0^2(a_1)) = u_1/2,$$

$$\lambda_1 \Re(\chi_0(a_3) - \chi_0(a_2)) + \lambda_2 \Re(\chi_0^2(a_3) - \chi_0^2(a_2)) = u_2/2,$$

and real numbers $\lambda_3$ and $\lambda_4$ such that

$$\lambda_3 \Im(\chi_0^h(a_2) - \chi_0^h(a_1)) + \lambda_4 \Im(\chi_0^k(a_2) - \chi_0^k(a_1)) = v_1/2,$$

$$\lambda_3 \Im(\chi_0^h(a_3) - \chi_0^h(a_2)) + \lambda_4 \Im(\chi_0^k(a_3) - \chi_0^k(a_2)) = v_2/2,$$

By Lemma 4.1, the six characters $\chi_0, \chi_0^2, \chi_0^3, \overline{\chi}_0, \overline{\chi}_0^2, \overline{\chi}_0^3$ are distinct. Now set $\mu_\chi = \lambda_1$ for $\chi \in \{\chi_0, \overline{\chi}_0\}$, $\mu_\chi = \lambda_2$ for $\chi \in \{\chi_0^2, \overline{\chi}_0^2\}$, and $\mu_\chi = 0$ for other characters. Also, let $\nu_{\chi_0} = \lambda_3$, $\nu_{\overline{\chi}_0} = -\lambda_3$, $\nu_{\chi_0^2} = \lambda_4$, $\nu_{\overline{\chi}_0^2} = -\lambda_4$, and $\nu_\chi = 0$ for other characters. Let $\theta_\chi = \mu_\chi + \nu_\chi$ for each $\chi$. Then (4.3) holds with $\lambda_\chi = \theta_\chi$ for each $\chi$, but it may occur that $\theta_\chi < 0$ for some $\chi$. However, by Lemma 4.1, $a_j \not\equiv 1 \pmod{q}$ for each $j$, so $\sum_{\chi \in C_q} \chi(a_j) = -1$ for every $j$. Thus, for any real $y$, (4.3) holds with $\lambda_\chi = \theta_\chi + y$ for each $\chi$.

Lemma 4.4. If $a_1, a_2, a_3$ do not satisfy the conditions of Lemma 2.1, then for all $\tau > 0$ and $\sigma > \frac{1}{2}$, there is a finite barrier for $D = (q, a_1, a_2, a_3)$, with each $B(\chi)$ consisting of numbers $\rho$ with $\Re \rho \leq \sigma$ and $\Im \rho > \tau$.

Proof. By Lemma 4.3, we can find such nonnegative $\nu_\chi^{(1)}$ and $\nu_\chi^{(2)}$ that

$$i = \sum_{\chi} \nu_\chi^{(1)}(\overline{\chi}(a_2) - \overline{\chi}(a_1)),$$

$$-i = \sum_{\chi} \nu_\chi^{(1)}(\overline{\chi}(a_3) - \overline{\chi}(a_2)),$$

$$i = \sum_{\chi} \nu_\chi^{(2)}(\overline{\chi}(a_2) - \overline{\chi}(a_1)),$$

$$i = \sum_{\chi} \nu_\chi^{(2)}(\overline{\chi}(a_3) - \overline{\chi}(a_2)).$$

(4.4)
Fix small positive \( \varepsilon > 0 \) and take a positive integer \( Q \) and nonnegative integers \( N^{(1)}_\chi \), \( N^{(2)}_\chi \) for all characters \( \chi \) modulo \( q \) such that \( |\nu^{(1)}_\chi - N^{(1)}_\chi/Q| < \varepsilon \), \( |\nu^{(2)}_\chi - N^{(2)}_\chi/Q| < \varepsilon \).

For some \( \sigma_1 \in (\beta_1, \sigma] \) and large \( \gamma > \tau \), suppose that for all characters \( \chi \in C_q \) and for \( k = 1, 2 \) the function \( L(s, \chi) \) has a zero at \( s = \sigma_1 + ki\gamma \) of order \( N^{(k)}_\chi \).

Suppose all other non-trivial zeros of \( L\)-functions modulo \( q \) have real part \( \leq \beta_1 \).

Let \( D_1(x) = \phi(q)(\pi_{q,a_1}(x) - \pi_{q,a_2}(x)) \) and \( D_2(x) = \phi(q)(\pi_{q,a_2}(x) - \pi_{q,a_3}(x)) \). By Lemma 1.1 and (4.4), we have

\[
\frac{\log x}{x^{\sigma_1}} D_1(x) = \frac{Q}{2\gamma} (2 \cos(\gamma \log x) + \cos(2\gamma \log x) + \varepsilon_1(x) + O(1/\gamma)),
\]

\[
\frac{\log x}{x^{\sigma_1}} D_2(x) = \frac{Q}{2\gamma} (-2 \cos(\gamma \log x) + \cos(2\gamma \log x) + \varepsilon_2(x) + O(1/\gamma)),
\]

where the functions \( \varepsilon_1(x) \), \( \varepsilon_2(x) \) are uniformly small if \( \varepsilon \) is small. Taking into account that \( \min(2 \cos u + \cos 2u, -2 \cos u + \cos 2u) \leq -1 \) for all \( u \), we obtain that for large \( x \), \( \pi_{q,a_1}(x) > \pi_{q,a_2}(x) > \pi_{q,a_3}(x) \) does not occur. □

5. A BARRIER SATISFYING GSH\(_q\)

The construction of this barrier is modeled on the construction in §2. For one character, \( B(\chi) \) is infinite, the number of elements of \( B(\chi) \) with imaginary part \( \leq T \) growing like \( \sqrt{T} \). By altering the parameters in the construction, we can create barriers with \( \sqrt{T} \) replaced by \( T^\epsilon \) for any fixed \( \epsilon \). Assume that for some relabeling of \( a_1, a_2, a_3 \), there are two characters \( \chi_1, \chi_2 \) satisfying

\[(5.1) \quad \chi_1(a_1) = \chi_1(a_2) \neq \chi_1(a_3), \quad \chi_2(a_1) \neq \chi_2(a_2).\]

Suppose that \( \frac{1}{2} \leq \beta < \sigma_2 < \sigma_1 \), that \( t \) is large and that \( L(s, \chi_1) \) has a simple zero at \( s = \sigma_1 + it \). Suppose that \( L(s, \chi_2) \) has simple zeros at the points \( s = \rho_j \) \((j = 1, 2, \ldots)\), where \( \rho_j = \sigma_2 - \delta_j + i\gamma_j \), \( \delta_j > 0 \), \( \gamma_j > 0 \), \( \delta_j \to 0 \) and \( \gamma_j \to \infty \) as \( j \to \infty \), and \( \sum 1/\gamma_j < \infty \). Also, suppose the numbers \( t, \gamma_1, \gamma_2, \ldots \) are linearly independent over \( \mathbb{Q} \).

Define

\[Z = \overline{\chi_1(a_2)} - \overline{\chi_1(a_3)}, \quad W = \overline{\chi_2(a_2)} - \overline{\chi_2(a_1)}.\]

By (5.1), \( Z \neq 0 \) and \( W \neq 0 \). Also define

\[\alpha = -\frac{1}{\pi} \left( \tan^{-1} \frac{\sigma_1}{t} + \arg Z \right), \quad \beta = \frac{\arg W}{2\pi} - \frac{1}{4}.
\]

Let \( H \) be the set of integers \( h \) such that \( \|h\alpha + \beta\| \leq \frac{1}{5} \). Since the number of possibilities for \( Z \) is finite, if \( t \) is large then

\[\frac{1}{10t} \leq \|\alpha\| \leq \frac{1}{2} - \frac{1}{10t}.
\]
It follows that in every set of \([10t] + 1\) consecutive integers, one of them is in \(H\). As in section 2, define

\[
D_1(x) := \phi(q)(\pi_{q,a_1}(x) - \pi_{q,a_2}(x)), \quad D_2(x) := \phi(q)(\pi_{q,a_3}(x) - \pi_{q,a_2}(x)).
\]

Suppose \(x\) is sufficiently large, and for brevity write \(u = \log x\). By Lemma 1.1 and our hypotheses,

\[
D_2(x) = \frac{2x^{\sigma_1}}{u} \left[ \Re \left( \frac{e^{itu}}{\sigma_1 + it} Z \right) + O \left( \frac{1}{u} \right) \right]
\]

and

\[
D_1(x) = \frac{2x^{\sigma_2}}{u} \sum_{\gamma_j \leq x} \left[ \Re \left( \frac{e^{(-\delta_j + i\gamma_j)u}}{\sigma_2 - \delta_j + i\gamma_j} W \right) + O \left( \frac{e^{-\delta_j u}}{\gamma_j^2 u} \right) \right] + O(x^\beta \log^2 x)
\]

\[
= \frac{2x^{\sigma_2}}{u} \sum_{\gamma_j \leq x} \left[ \Re B_j + O \left( \frac{e^{-\delta_j u}}{\gamma_j^2} \right) \right] + O(x^\beta \log^2 x),
\]

where

\[
B_j = W \frac{e^{(-\delta_j + i\gamma_j)u}}{i\gamma_j}.
\]

By assumption, \(\sum_{j} |B_j| \ll 1\), thus \(D_1(x) \ll x^{\sigma_2}/u\). Modulo \(2\pi\),

\[
\arg \frac{e^{itu}}{\sigma_1 + it} Z \equiv tu - \tan^{-1} \frac{t}{\sigma_1} + \arg Z \equiv tu - \frac{\pi}{2} - \pi\alpha.
\]

By (5.2), when \(\|tu/\pi - \alpha\| \geq u^{-0.9}\), \(D_2(x) \gg x^{\sigma_1}/(\log x)^{1.9}\), and thus for these \(x\) either \(\pi_{q,a_3}(x)\) is the largest or smallest of the three functions. Next assume that

\[
\|tu/\pi - \alpha\| \leq u^{-0.9}.
\]

We choose \(\delta_j\) and \(\gamma_j\) as follows: \(0 < \delta_j < \sigma_2 - \beta, j^{-3} \ll \delta_j \ll j^{-3}, \gamma_j = 2th_j + O(j^{-10})\), where for \(j \geq 10t\) we have \(h_j \in H, h_{j+1} > h_j\) and \(j^2 \leq h_j \leq j^2 + j\). With these choices,

\[
\sum_{j=1}^{\infty} \frac{e^{-\delta_j u}}{\gamma_j^2} \ll e^{-u^{1/4}} \sum_{j \leq u^{1/4}} 1/j^4 + \sum_{j > u^{1/4}} 1/j^4 \ll u^{-3/4}
\]

and

\[
\sum_{j < u^{1/4} \text{ or } j > u^{2/5}} \frac{e^{-\delta_j u}}{\gamma_j} \ll e^{-u^{1/4}} + u^{-2/5} \ll u^{-2/5}.
\]
Thus, by (5.3) and (5.4),

\begin{equation}
D_1(x) = \frac{2x^{\sigma_2}}{u} \left[ \sum_{u^{1/4} \leq j \leq u^{2/5}} \Re B_j + O(u^{-0.4}) \right].
\end{equation}

Suppose \( u^{1/4} \leq j \leq u^{2/5} \). Since \( h_j \in H \), we have

\[
\left\| \frac{1}{2\pi} \arg B_j \right\| = \left\| \frac{1}{2\pi} \left( \arg W + \gamma_j u - \frac{\pi}{2} \right) \right\| = \left\| \beta + \frac{u t}{\pi} h_j + O(u^{-3/2}) \right\| = \left\| \beta + h_j \alpha + O(u^{-0.1}) \right\| \leq 0.21
\]

for large \( u \). Hence \( \Re B_j \geq |B_j| \cos(0.42\pi) \geq \frac{1}{5}|B_j| \). Therefore,

\[
\sum_{u^{1/4} \leq j \leq u^{2/5}} \Re B_j \gg \sum_{u^{1/3} \leq j \leq 2u^{1/3}} \frac{1}{\gamma_j} \gg u^{-1/3}.
\]

It follows from (5.5) that for \( u \) large and \( \| \frac{u t}{\pi} - \alpha \| \leq u^{-0.9} \) that

\[
D_1(x) \geq \frac{c x^{\sigma_2}}{(\log x)^{4/3}}
\]

where \( c > 0 \) depends on \( q, t \) and \( W \). This implies that the inequality \( \pi_{q,a_2}(x) > \pi_{q,a_3}(x) > \pi_{q,a_1}(x) \) does not occur for large \( x \).

REFERENCES

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