

CHEBYSHEV'S BIAS FOR PRODUCTS OF TWO PRIMES

KEVIN FORD AND JASON SNEED

ABSTRACT. Under two assumptions, we determine the distribution of the difference between two functions each counting the numbers $\leq x$ that are in a given arithmetic progression modulo q and the product of two primes. The two assumptions are (i) the Extended Riemann Hypothesis for Dirichlet L -functions modulo q , and (ii) that the imaginary parts of the nontrivial zeros of these L -functions are linearly independent over the rationals. Our results are analogs of similar results proved for primes in arithmetic progressions by Rubinstein and Sarnak.

1. INTRODUCTION

1.1. Prime number races. Let $\pi(x; q, a)$ denote the number of primes in the progression $a \pmod q$. For fixed q , the functions $\pi(x; q, a)$ (for $a \in A_q$, the set of residues coprime to q) all satisfy

$$(1.1) \quad \pi(x, q, a) \sim \frac{x}{\varphi(q) \log x},$$

where φ is Euler's totient function [Da]. There are, however, curious inequities. For example $\pi(x; 4, 3) \geq \pi(x; 4, 1)$ seems to hold for most x , an observation of Chebyshev from 1853 [Ch]. In fact, $\pi(x; 4, 3) < \pi(x; 4, 1)$ for the first time at $x = 26,861$ [Le]. More generally, one can ask various questions about the behavior of

$$(1.2) \quad \Delta(x; q, a, b) := \pi(x; q, a) - \pi(x; q, b)$$

for distinct $a, b \in A_q$. Does $\Delta(x; q, a, b)$ change sign infinitely often? Where is the first sign change? How many sign changes with $x \leq X$? What are the extreme values of $\Delta(x; q, a, b)$? Such questions are colloquially known as *prime race problems*, and were studied extensively by Knapowski and Turán in a series of papers beginning with [KT]. See the survey articles [FK] and [GM] and references therein for an introduction to the subject and summary of major findings. Properties of Dirichlet L -functions lie at the heart of such investigations.

Despite the tendency for the function $\Delta(x; 4, 3, 1)$ to be negative, Littlewood [Li] showed that it changes sign infinitely often. Similar results have been proved for other q, a, b (see [S] and references therein). Still, in light of Chebyshev's observation, we can ask how frequently $\Delta(x; q, a, b)$ is positive and how often it is negative. These questions are best addressed in the context of *logarithmic density*. A set S of positive integers has logarithmic density

$$\delta(S) = \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{\substack{n \leq x \\ n \in S}} \frac{1}{n}$$

provided the limit exists. Let $\delta(q, a, b) = \delta(P(q, a, b))$, where $P(q, a, b)$ is the set of integers n with $\Delta(n; q, a, b) > 0$. In 1994, Rubinstein and Sarnak [RS] showed that $\delta(q; a, b)$ exists, assuming two hypotheses (i) the Extended Riemann Hypothesis for Dirichlet L -functions modulo q (ERH_q), and (ii) the imaginary parts of zeros of each Dirichlet L -function are linearly independent over the rationals (GSH_q - Grand Simplicity Hypothesis). The authors also gave methods to accurately estimate the "bias", for example showing that $\delta(4; 3, 1) \approx 0.996$ in Chebyshev's case. More generally, $\delta(q; a, b) = \frac{1}{2}$ when a and b are either

Date: August 3, 2009.

2000 Mathematics Subject Classification: 11M06, 11N13, 11N25.

The research of K. F. was supported in part by National Science Foundation grants DMS-0555367 and DMS-0901339.

both quadratic residues modulo q or both quadratic nonresidues (unbiased prime races), but $\delta(q; a, b) > \frac{1}{2}$ whenever a is a quadratic non-residue and b is a quadratic residue. A bit later we will discuss the reasons behind these phenomena.

1.2. Quasi-prime races. In this paper we develop a parallel theory for comparison of functions $\pi_2(x; q, a)$, the number of integers $\leq x$ which are in the progression $a \pmod q$ and which are the product of two primes $p_1 p_2$ ($p_1 = p_2$ allowed). Put

$$\Delta_2(x; q, a, b) := \pi_2(x; q, a) - \pi_2(x; q, b),$$

let $P_2(q, a, b)$ be the set of integers n with $\Delta_2(n; q, a, b) > 0$, and set $\delta_2(q, a, b) = \delta(P_2(q, a, b))$. The table below shows all such quasi-primes up to 100 grouped in residue classes modulo 4.

$pq \equiv 1 \pmod{4}$	$pq \equiv 3 \pmod{4}$
9	15
21	35
25	39
33	51
49	55
57	87
65	91
69	95
77	
85	
93	

Observe that $\Delta_2(x; 4, 3, 1) \leq 0$ for $x \leq 100$, and in fact the smallest x with $\Delta_2(x; 4, 3, 1) > 0$ is $x = 26747$ (amazingly close to the first sign change of $\Delta(x; 4, 3, 1)$). Some years ago Richard Hudson conjectured that the bias for products of two primes is always reversed from that of primes; i.e., $\delta_2(q; a, b) < \frac{1}{2}$ when a is a quadratic non-residue modulo q and b is a quadratic residue. Under the same assumptions as [RS], namely ERH_q and GSH_q , we confirm Hudson's conjecture and also show that the bias is less pronounced.

Theorem 1. *Let a, b be distinct elements of A_q . Assuming ERH_q and GSH_q , $\delta_2(q; a, b)$ exists. Moreover, if a and b are both quadratic residues modulo q or both quadratic non-residues, then $\delta_2(q; a, b) = \frac{1}{2}$. Otherwise, if a is a quadratic nonresidue and b is a quadratic residue, then*

$$1 - \delta(q; a, b) < \delta_2(q; a, b) < \frac{1}{2}.$$

We can accurately estimate $\delta_2(q; a, b)$ borrowing methods from [RS, §4]. In particular we have

$$\delta_2(4; 3, 1) \approx 0.10572.$$

We deduce Theorem 1 by connecting the distribution of $\Delta_2(x; q, a, a)$ with the distribution of $\Delta(x; q, a, b)$. Although the relationship is “simple”, there is no elementary way to derive it, say by writing

$$\pi_2(x; q, a) = \frac{1}{2} \sum_{p \leq x} \pi(x; q, ap^{-1} \pmod q) + \frac{1}{2} \sum_{\substack{p \leq \sqrt{x} \\ p^2 \equiv a \pmod q}} 1.$$

In particular, our result depends strongly on the assumption that the zeros of the L -functions modulo q have only simple zeros. Let $N(q, a)$ be the number of $x \in A_q$ with $x^2 \equiv a \pmod q$, and let $C(q)$ be the set of nonprincipal Dirichlet characters modulo q .

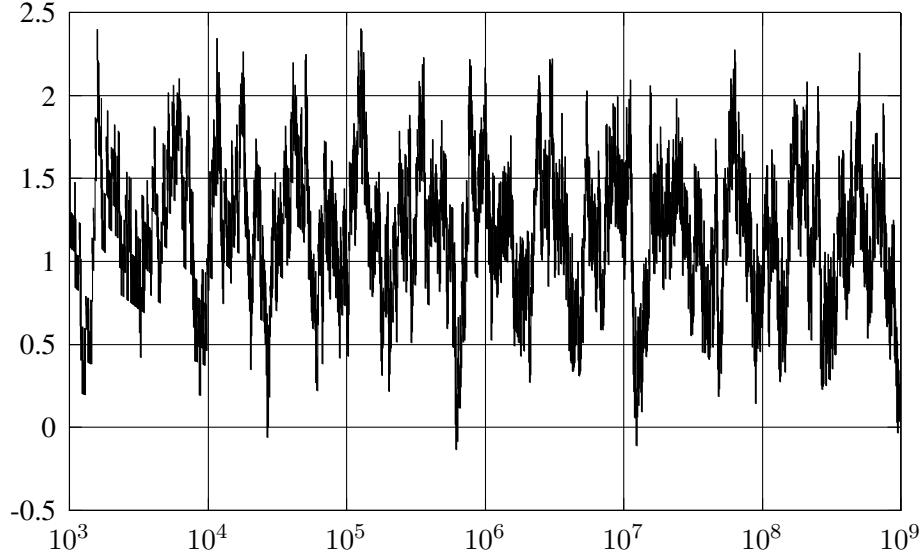


FIGURE 1. $\frac{\log x}{\sqrt{x}} \Delta(x; 4, 3, 1)$

Theorem 2. Assume ERH_q and for each $\chi \in C(q)$, $L(\frac{1}{2}, \chi) \neq 0$ and the zeros of $L(s, \chi)$ are simple. Then

$$\frac{\Delta_2(x; q, a, b) \log x}{\sqrt{x} \log \log x} = \frac{N(q, b) - N(q, a)}{2\phi(q)} - \frac{\log x}{\sqrt{x}} \Delta(x; q, a, b) + \Sigma(x; q, a, b),$$

where $\frac{1}{Y} \int_1^Y |\Sigma(e^y; q, a, b)|^2 dy = o(1)$ as $Y \rightarrow \infty$.

The expression for Δ_2 given in Theorem 2 must be modified if some $L(s, \chi)$ has multiple zeros; see §3 for details.

Figures 1,2 and 3 show graphs corresponding to $(q, a, b) = (4, 3, 1)$, plotted on a logarithmic scale from $x = 10^3$ to $x = 10^9$. While $\Sigma(x; 4, 3, 1)$ appears to be oscillating around -0.2 , this is caused by some terms in $\Sigma(x; 4, 3, 1)$ of order $1/\log \log x$, and $\log \log 10^9 \approx 3.03$. By Theorem 2, $\Sigma(x; 4, 3, 1)$ will (assuming ERH_4 and GSH_4) eventually settle down to oscillating about 0.

It is not immediate that Theorem 1 follows from Theorem 2. One first needs more precise information about the distribution of $\Delta(x; q, a, b)$ from [RS].

Theorem RS. [RS, §1] Assume ERH_q and GSH_q . For any distinct $a, b \in A_q$, the function

$$(1.3) \quad \frac{u\Delta(e^u; q, a, b)}{e^{u/2}}$$

has a probabilistic distribution. This distribution (i) has mean and median $(N(q, b) - N(q, a))/\phi(q)$, (ii) is symmetric with respect to its mean, and (iii) has a continuous density function.

Assume a is a quadratic nonresidue modulo q and b is a quadratic residue. Then $N(q, b) - N(q, a) > 0$. Let f be the density function for the distribution of (1.3), that is,

$$f(t) = \frac{d}{dt} \lim_{U \rightarrow \infty} \frac{1}{U} \text{meas}\{0 \leq u \leq U : ue^{-u/2} \Delta(e^u; q, a, b) \leq t\}.$$

We see from Theorems 2 and RS that

$$\delta(q, a, b) = \int_0^\infty f(t) dt > \frac{1}{2}$$

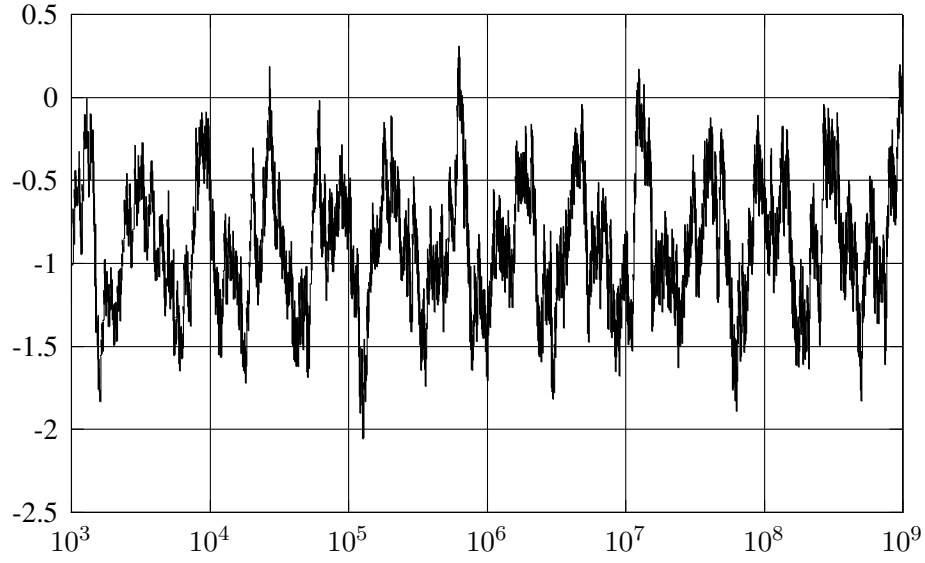


FIGURE 2. $\frac{\log x}{\sqrt{x} \log \log x} \Delta_2(x; 4, 3, 1)$

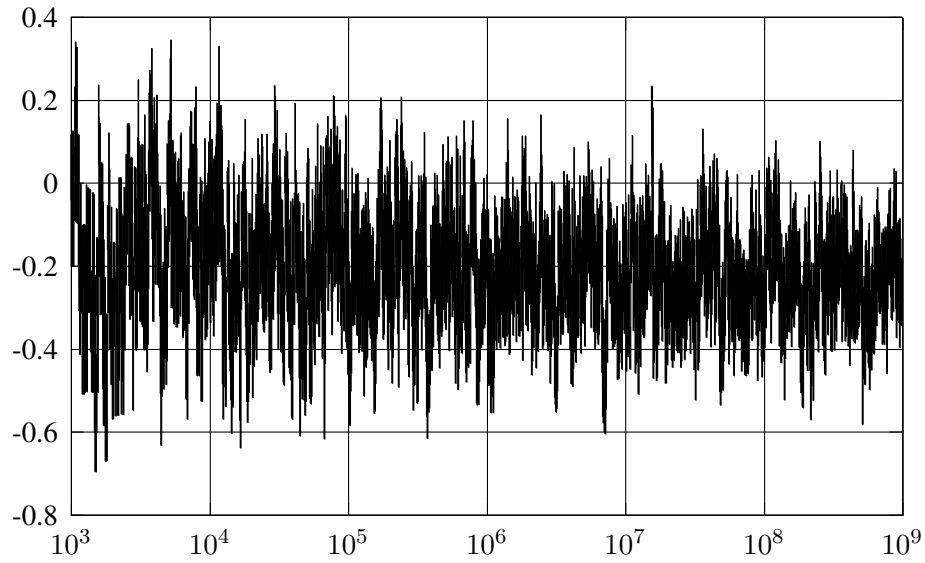


FIGURE 3. $\Sigma(x; 4, 3, 1)$

and

$$\delta_2(q, a, b) = \int_{-\infty}^{\frac{N(q,b) - N(q,a)}{2\phi(q)}} f(t) dt,$$

from which Theorem 1 follows.

Theorem 2 also determines the joint distribution of any vector function

$$(1.4) \quad \frac{u}{e^{u/2} \log u} (\Delta_2(e^u; q, a_1, b_1), \dots, \Delta_2(e^u; q, a_r, b_r)).$$

Theorem 3. *If $f(x_1, \dots, x_r)$ is the density function of*

$$\frac{u}{e^{u/2}} (\Delta(e^u; q, a_1, b_1), \dots, \Delta(e^u; q, a_r, b_r)),$$

then the joint density function of (1.4) is

$$f \left(\frac{N(q, b_1) - N(q, a_1)}{2\phi(q)} - x_1, \dots, \frac{N(q, b_r) - N(q, a_r)}{2\phi(q)} - x_r \right).$$

1.3. Origin of Chebyshev's bias. From an analytic point of view (L -functions), the weighted sum

$$(1.5) \quad \Delta^*(x; q, a, b) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n) - \sum_{\substack{n \leq x \\ n \equiv b \pmod{q}}} \Lambda(n),$$

where Λ is the von Mangoldt function, is more natural than (1.2). Expressing $\Delta^*(x; q, a, b)$ in terms of sums over zeros of L -functions in the standard way (§19 of [Da]), we obtain, on ERH_q ,

$$e^{-u/2} \phi(q) \Delta^*(e^u; q, a, b) = - \sum_{\chi \in C(q)} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\gamma} \frac{e^{i\gamma u}}{1/2 + i\gamma} + O(u^2 e^{-u/2}),$$

where γ runs over imaginary parts of nontrivial zeros of $L(s, \chi)$ (counted with multiplicity). Hypothesis GSH_q implies, in particular, that $L(1/2, \chi) \neq 0$. Each summand $e^{i\gamma u}/(1/2 + i\gamma)$ is thus a harmonic with mean zero as $u \rightarrow \infty$, and GSH_q implies that the harmonics behave independently. Hence, we expect that $e^{-u/2} \phi(q) \Delta^*(e^u; q, a, b)$ will behave like a mean zero random variable. On the other hand, the right side of (1.5) contains not only terms corresponding to prime n but terms corresponding to powers of primes. Applying the prime number theorem for arithmetic progressions (1.1) to the terms $n = p^2$ in (1.5) gives

$$\Delta^*(x; q, a, b) = \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p - \sum_{\substack{p \leq x \\ p \equiv b \pmod{q}}} \log p + \frac{x^{1/2}}{\phi(q)} (N(q, a) - N(q, b)) + O(x^{1/3}).$$

Hence, on ERH_q and GSH_q , we expect the expression

$$(1.6) \quad \frac{1}{\sqrt{x}} \left(\sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p - \sum_{\substack{p \leq x \\ p \equiv b \pmod{q}}} \log p \right)$$

to behave like a random variable with mean $(N(q, b) - N(q, a))/\phi(q)$. Finally, the distribution of $\Delta(x; q, a, b)$ is obtained from the distribution of (1.6) and partial summation.

1.4. Analyzing $\Delta_2(x; q, a, b)$. A natural analog of $\Delta^*(x; q, a, b)$ is

$$(1.7) \quad \sum_{\substack{mn \leq x \\ mn \equiv a \pmod{q}}} \Lambda(m)\Lambda(n) - \sum_{\substack{mn \leq x \\ mn \equiv b \pmod{q}}} \Lambda(m)\Lambda(n).$$

As with $\Delta^*(x; q, a, b)$, the expression in (1.7) can be easily written as a sum over zeros of L -functions plus a small error. The main problem now is that the principal summands, namely $\log p_1 \log p_2$ for primes p_1, p_2 , are very irregular as a function of $p_1 p_2$, and thus estimates for $\Delta_2(x; q, a, b)$ cannot be recovered by partial summation. We get around this problem using a double integration, a method which goes back to Landau [La, §88]. We have

$$(1.8) \quad \begin{aligned} \Delta_2(x; q, a, b) &= \frac{1}{\phi(q)} \sum_{\chi \in C(q)} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\substack{n = p_1 p_2 \leq x \\ p_1 \leq p_2}} \chi(n) \\ &= \frac{1}{2\phi(q)} \sum_{\chi \in C(q)} (\bar{\chi}(a) - \bar{\chi}(b)) \int_0^\infty \int_0^\infty G(x, u, v; \chi) du dv + O\left(\frac{\sqrt{x}}{\log x}\right), \end{aligned}$$

where

$$(1.9) \quad G(x, u, v; \chi) = \sum_{p_1 p_2 \leq x} \frac{\chi(p_1 p_2) \log p_1 \log p_2}{p_1^u p_2^v}.$$

The related functions

$$G^*(x, u, v; \chi) = \sum_{mn \leq x} \frac{\chi(mn) \Lambda(m) \Lambda(n)}{m^u n^v}$$

are more “natural” from an analytic point of view, being easily expressed in terms of zeros of Dirichlet L -functions. By the reasoning of the previous subsection, each $G^*(x, u, v; \chi)$ is expected to be unbiased, the bias in $\Delta_2(x; q, a, b)$ originating from the summands in $G^*(x, u, v; \chi)$ where m is not prime or n is not prime.

1.5. A heuristic argument for the bias in $\Delta_2(x; q, a, b)$. We conclude this introduction with an heuristic evaluation of the bias in $\Delta_2(x; q, a, b)$, which originates from the difference between functions $G(x; u, v; \chi)$ and $G^*(x, u, v; \chi)$. For simplicity of exposition, we’ll concentrate on the special case $(q, a, b) = (4, 3, 1)$. Let χ be the non-principal character modulo 4. The bias comes from

$$\frac{1}{2} \int_0^\infty \int_0^\infty (G^*(x, u, v; \chi) - G(x, u, v; \chi)) du dv = \frac{1}{2} \sum_{\substack{p_1^a p_2^b \leq x \\ \max(a, b) \geq 2}} \frac{\chi(p_1^a p_2^b)}{ab}.$$

There are $O(x^{1/2}/\log x)$ terms with $\min(a, b) \geq 2$ and $\max(a, b) \geq 3$. By the prime number theorem and partial summation,

$$\frac{1}{2} \sum_{p_1^2 p_2^2 \leq x} \frac{1}{4} = \frac{1}{8} \sum_{p \leq \sqrt{x}} \pi(\sqrt{x/p^2}) \sim \frac{x^{1/2} \log \log x}{2 \log x}.$$

Thus,

$$\begin{aligned} \Delta_2(x; 4, 3, 1) &= -\frac{1}{2} \sum_{mn \leq x} \frac{\chi(mn) \Lambda(m) \Lambda(n)}{\log m \log n} - \left(\sum_{k=2}^\infty \frac{1}{k} \sum_{p_1^k \leq x} \chi(p_1^k) \Delta(x/p_1^k; 4, 3, 1) \right) \\ &\quad + \left(\frac{1}{2} + o(1) \right) \frac{x^{1/2} \log \log x}{\log x}. \end{aligned}$$

By Theorem RS, $\Delta(y; 4, 3, 1) = y^{1/2}/\log y + E(y)$, where $E(y)$ oscillates with mean 0. Thus,

$$\sum_{k=2}^\infty \frac{1}{k} \sum_{p_1^k \leq x} \chi(p_1^k) \Delta(x/p_1^k; 4, 3, 1) = \sum_{k=2}^\infty \frac{2}{k} \sum_{p_1^k \leq x} \chi(p_1^k) \frac{\sqrt{x/p_1^k}}{\log(x/p_1^k)} + E'(x),$$

where $E'(x)$ is expected to oscillate with mean zero. The $k = 2$ terms are

$$\sum_{p_1^2 \leq x} \frac{\sqrt{x/p_1^2}}{\log(x/p_1^2)} \sim \frac{\sqrt{x} \log \log x}{\log x},$$

while the terms corresponding to $k \geq 3$ contribute

$$\ll \sum_{k=3}^\infty \frac{1}{k} \sum_{p_1^k \leq x} \frac{\sqrt{x/p_1^k}}{\log(x/p_1^k)} \ll \frac{\sqrt{x}}{\log x}.$$

Thus, we find that

$$\Delta_2(x; 4, 3, 1) = -\frac{1}{2} \sum_{mn \leq x} \frac{\chi(mn) \Lambda(m) \Lambda(n)}{\log m \log n} - \left(\frac{1}{2} + o(1) \right) \frac{x^{1/2} \log \log x}{\log x} + E'(x).$$

Observe that the bias arises from terms $p_1 p_2^2$ and $p_1^2 p_2^2$ which appear in $G^*(x; u, v; \chi)$ but not in $G(x, u, v; \chi)$.

1.6. Further problems. It is natural to consider the distribution, in arithmetic progressions, of numbers composed of exactly k prime factors, where $k \geq 3$ is fixed. As with the cases $k = 1$ and $k = 2$, we expect there to be no bias if we count all numbers $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ with weight $(a_1 \cdots a_k)^{-1}$. If, however, we count terms which are the product of precisely k primes (that is, numbers $p_1^{a_1} \cdots p_j^{a_j}$ with $a_1 + \cdots + a_j = k$), then there will be a bias. Hudson has conjectured that the bias will be in the same direction as for primes when k is odd, and in the opposite direction for even k . We conjecture that, in addition, the bias becomes less pronounced as k increases.

2. PRELIMINARIES

With χ fixed, the letter γ , with or without subscripts, denotes the imaginary part of a zero of $L(s, \chi)$ inside the critical strip. In sums over γ , each term appears with its multiplicity $m(\gamma)$ unless we specify that we sum over distinct γ . Constants implied by O - and \ll -symbols depend only on χ (and hence, on q) unless additional dependence is indicated with a subscript. Let

$$A(\chi) = \begin{cases} 1 & \chi^2 = \chi_0 \\ 0 & \text{else} \end{cases},$$

where χ_0 is the principal character modulo q . That is, $A(\chi) = 1$ if and only if χ is a real character. For $\chi \in C(q)$, define

$$F(s, \chi) = \sum_p \frac{\chi(p) \log p}{p^s}.$$

The following estimates are standard; see e.g. [Da, §15,16].

Lemma 2.1. *Let $\chi \in C(q)$, assume ERH_q and fix $c > \frac{1}{3}$. Then $F(s, \chi) = -\frac{L'}{L}(s, \chi) + A(\chi) \frac{\zeta'}{\zeta}(2s) + H(s, \chi)$, where $H(s, \chi)$ is analytic and uniformly bounded in the half-plane $\Re s \geq c$.*

Lemma 2.2. *Let χ be a Dirichlet character modulo q . Let $N(T, \chi)$ denote the number of zeros of $L(s, \chi)$ with $0 < \Re s < 1$ and $|\Im s| < T$. Then*

- (1) $N(T, \chi) = O(T \log(qT))$ for $T \geq 1$.
- (2) $N(T, \chi) - N(T-1, \chi) = O(\log(qT))$ for $T \geq 1$.
- (3) Uniformly for $s = \sigma + it$ and $\sigma \geq -1$,

$$\frac{L'(s, \chi)}{L(s, \chi)} = \sum_{|\gamma-t| < 1} \frac{1}{s - \rho} + O(\log q(|t| + 2)).$$

- (4) $-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1)$ uniformly for $\sigma \geq \frac{1}{2}$, $\sigma \neq 1$.
- (5) $|\frac{\zeta'}{\zeta}(\sigma + iT)| \leq -\frac{\zeta'}{\zeta}(\sigma)$ for $\sigma > 1$.

For a suitably small, fixed $\delta > 0$, we say that a number $T \geq 2$ is *admissible* if for all $\chi \in C(q) \cup \{\chi_0\}$ and all zeros $\frac{1}{2} + i\gamma$ of $L(s, \chi)$, $|\gamma - T| \geq \delta(\log T)^{-1}$. By Lemma 2.2, we can choose δ small enough, depending on q , so that there is an admissible T in $[U, U+1]$ for all $U \geq 2$. From Lemma 2.2 we obtain

Lemma 2.3. *Uniformly for $\sigma \geq \frac{2}{5}$ and admissible $T \geq 2$,*

$$|F(\sigma + iT, \chi)| = O(\log^2 T).$$

Lemma 2.4. *Fix $\chi \in C(q)$ and assume $L(\frac{1}{2}, \chi) \neq 0$. For $A \geq 0$ and real $k \geq 0$,*

$$\sum_{\substack{|\gamma_1|, |\gamma_2| \geq A \\ |\gamma_1 - \gamma_2| \geq 1}} \frac{\log^k(|\gamma_1| + 3) \log^k(|\gamma_2| + 3)}{|\gamma_1| |\gamma_2| |\gamma_1 - \gamma_2|} \ll_k \frac{(\log(A+3))^{2k+3}}{A+1}.$$

Proof. The sum in question is at most twice the sum of terms with $|\gamma_2| \geq |\gamma_1|$, which is

$$\ll \sum_{|\gamma_2| \geq A} \frac{\log^{2k}(|\gamma_2| + 3)}{|\gamma_2|} \left(\frac{1}{|\gamma_2|} \sum_{|\gamma_1| < \frac{|\gamma_2|}{2}} \frac{1}{|\gamma_1|} + \frac{1}{|\gamma_2|} \sum_{\substack{|\frac{\gamma_2|}{2} \leq |\gamma_1| \leq |\gamma_2| \\ |\gamma_2 - \gamma_1| \geq 1}} \frac{1}{|\gamma_2 - \gamma_1|} \right).$$

By Lemma 2.2 (1), the two sums over γ_1 are $O(\log^2(|\gamma_2| + 3))$. A further application of Lemma 2.2 (1) completes the proof. \square

We conclude this section with a truncated version of the Perron formula for $G(x, u, v; \chi)$.

Lemma 2.5. *Uniformly for $x \leq T \leq 2x^2$, $x \geq 2$, $u \geq 0$ and $v \geq 0$, we have*

$$(2.1) \quad G(x, u, v; \chi) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s+u, \chi) F(s+v, \chi) \frac{x^s}{s} ds + O(\log^3 x),$$

where $c = 1 + \frac{1}{\log x}$.

Proof. For $\Re s > 1$, we have

$$F(s+u, \chi) F(s+v, \chi) = \sum_{n=1}^{\infty} f(n) n^{-s}, \quad f(n) = \sum_{p_1 p_2 = n} \frac{\chi(p_1 p_2) \log p_1 \log p_2}{p_1^u p_2^v}.$$

Using the trivial estimate $|f(n)| \leq \log^2 n$ and a standard argument [Da, §17, (3) and (5)], we obtain the desired bounds. \square

3. OUTLINE OF THE PROOF OF THEOREM 2

Throughout the remainder of this paper, fix q , assume ERH_q and that $L(\frac{1}{2}, \chi) \neq 0$ for each $\chi \in C(q)$. Let

$$\varepsilon = \frac{1}{100}.$$

We next define a function $T(x)$ as follows. For each positive integer n , let T_n be an admissible value of T satisfying $\exp(2^{n+1}) \leq T_n \leq \exp(2^{n+1}) + 1$ and set $T(x) = T_n$ for $\exp(2^n) < x \leq \exp(2^{n+1})$. In particular, we have

$$x \leq T(x) \leq 2x^2 \quad (x \geq e^2).$$

Our first task is to express the double integrals in (1.8) in terms of sums over zeros of $L(s, \chi)$. This is proved in Section 4.

Lemma 3.1. *Let $\chi \in C(q)$ and let $T = T(x)$. Then*

$$\begin{aligned} & x^{-1/2} \int_0^\infty \int_0^\infty G(x, u, v; \chi) du dv \\ &= 2 \int_0^{2\varepsilon} \int_0^{2\varepsilon} \sum_{|\gamma| \leq T} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi) x^{-v+i\gamma}}{\frac{1}{2} - v + i\gamma} du dv + \frac{A(\chi) \log \log x + \Sigma_1(x; \chi) + O(1)}{\log x}, \end{aligned}$$

where $\int_1^Y |\Sigma_1(e^y; \chi)|^2 dy = O(Y)$.

The double integral on the right side in Lemma 3.1 is complicated to analyze. In Section 5 we prove the following.

Lemma 3.2. *Let $\chi \in C(q)$. Let n be a positive integer, $2^n < \log x \leq 2^{n+1}$ and $T = T(x)$. Then*

$$2 \int_0^{2\varepsilon} \int_0^{2\varepsilon} \sum_{\substack{|\gamma| \leq T \\ \gamma \text{ distinct}}} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi) x^{-v+i\gamma}}{\frac{1}{2} - v + i\gamma} du dv = \frac{\Sigma_2(x; \chi)}{\log x} \\ + 2 \sum_{\substack{|\gamma| \leq T \\ \gamma \text{ distinct}}} m^2(\gamma) x^{i\gamma} \left(\frac{1}{2} + i\gamma\right) \int_0^{2\varepsilon-2^{-n}} \frac{x^{-v}}{\frac{1}{2} - v + i\gamma} \int_{v+2^{-n}}^{2\varepsilon} \frac{du}{(u-v)(\frac{1}{2} - u + i\gamma)} dv,$$

where $\int_1^Y |\Sigma_2(e^y; \chi)|^2 dy = o(Y \log^2 Y)$.

The terms on the right in Lemma 3.2 with small $|\gamma|$ will give the main term, and terms with larger $|\gamma|$ are considered as error terms. The next lemma is proved in Section 6.

Lemma 3.3. *Let $\chi \in C(q)$. Let n be a positive integer, $2^n < \log x \leq 2^{n+1}$, $T = T(x)$ and $2 \leq T_0 \leq T$. Then*

$$2 \sum_{\substack{|\gamma| \leq T \\ \gamma \text{ distinct}}} m^2(\gamma) x^{i\gamma} \left(\frac{1}{2} + i\gamma\right) \int_0^{2\varepsilon-2^{-n}} \frac{x^{-v}}{\frac{1}{2} - v + i\gamma} \int_{v+2^{-n}}^{2\varepsilon} \frac{du}{(u-v)(\frac{1}{2} - u + i\gamma)} dv \\ = \frac{2 \log \log x}{\log x} \sum_{\substack{|\gamma| \leq T_0 \\ \gamma \text{ distinct}}} \frac{m^2(\gamma) x^{i\gamma}}{1/2 + i\gamma} + O\left(\frac{\log^3 T_0}{\log x}\right) + \frac{\Sigma_3(x, T_0; \chi)}{\log x},$$

where

$$\frac{1}{Y} \int_1^Y |\Sigma_3(e^y, T_0; \chi)|^2 dy \ll \frac{\log^5 T_0}{T_0} \log^2 Y.$$

Combining Lemmas 3.1, 3.2 and 3.3 with (1.8) yields (for fixed, large T_0)

$$\Delta_2(x; q, a, b) = \frac{\sqrt{x}}{2\phi(q)} \sum_{\chi \in C(q)} (\bar{\chi}(a) - \bar{\chi}(b)) \left[\frac{\log \log x}{\log x} \left(A(\chi) + 2 \sum_{\substack{|\gamma| \leq T_0 \\ \gamma \text{ distinct}}} \frac{m^2(\gamma) x^{i\gamma}}{1/2 + i\gamma} \right) \right. \\ \left. + \frac{\Sigma_1(x; \chi) + \Sigma_2(x; \chi) + \Sigma_3(x, T_0; \chi) + O(\log^3 T_0)}{\log x} \right],$$

where

$$\lim_{T_0 \rightarrow \infty} \left(\limsup_{Y \rightarrow \infty} \frac{1}{Y \log^2 Y} \sum_{\chi \in C(q)} \int_1^Y |\Sigma_1(e^y; \chi) + \Sigma_2(e^y; \chi) + \Sigma_3(e^y; T_0; \chi)|^2 dy \right) = 0.$$

On the other hand (cf. [RS]),

$$\Delta(x; q, a, b) = \frac{\sqrt{x}}{\log x} \left(\frac{N(q, b) - N(q, a)}{\phi(q)} - \sum_{\chi \in C(q)} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{|\gamma| \leq T_0} \frac{x^{i\gamma}}{1/2 + i\gamma} + \Sigma_4(x; T_0) \right),$$

where

$$\lim_{T_0 \rightarrow \infty} \left(\limsup_{Y \rightarrow \infty} Y^{-1} \int_1^Y |\Sigma_4(e^y; T_0)|^2 dy \right) = 0.$$

Now assume $m(\gamma) = 1$ for all γ , and note that

$$\sum_{\chi \in C(q)} (\bar{\chi}(a) - \bar{\chi}(b)) A(\chi) = N(q, a) - N(q, b).$$

Letting $T_0 \rightarrow \infty$ finishes the proof of Theorem 2.

4. PROOF OF LEMMA 3.1

Assume ERH_q throughout. We first estimate $G(x, u, v; \chi)$ for different ranges of u, v .

Lemma 4.1. *Let $\chi \in C(q)$, $\chi \neq \chi_0$. For $x \geq 4$, the following hold:*

- (1) For $u, v \geq \varepsilon$, $G(x, u, v; \chi) \ll x^{\frac{1}{2}-\frac{\varepsilon}{2}} \log^5 x$.
(2) For $u \geq 2\varepsilon$, $v \leq \varepsilon$ and $T = T(x)$,

$$x^{-1/2}G(x, u, v; \chi) = \sum_{|\gamma| \leq T} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi)x^{-v+i\gamma}}{\frac{1}{2} - v + i\gamma} - A(\chi) \frac{F(\frac{1}{2} + u - v, \chi)x^{-v}}{1 - 2v} + O(x^{-\frac{3\varepsilon}{2}} \log^5 x).$$

- (3) For $u \leq 2\varepsilon$, $v \leq 2\varepsilon$, $u \neq v$ and $T = T(x)$,

$$x^{-1/2}G(x, u, v; \chi) = \sum_{|\gamma| \leq T} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi)x^{-v+i\gamma}}{\frac{1}{2} - v + i\gamma} + \frac{F(\frac{1}{2} - u + v + i\gamma, \chi)x^{-u+i\gamma}}{\frac{1}{2} - u + i\gamma} - A(\chi) \left(\frac{F(\frac{1}{2} + u - v, \chi)x^{-v}}{1 - 2v} + \frac{F(\frac{1}{2} - u + v, \chi)x^{-u}}{1 - 2u} \right) + O(x^{-3\varepsilon} \log^5 x).$$

Proof. Assume $u \geq \varepsilon$ and $v \geq \varepsilon$. Start with the approximation of $G(x, u, v; \chi)$ given by Lemma 2.5, then deform the segment of integration to the contour consisting of three straight segments connecting $c - iT$, $b - iT$, $b + iT$ and $c + iT$, where $b = \frac{1}{2} - \frac{\varepsilon}{2}$ and $T = T(x)$. The rectangle formed by the new and old contours does not contain any poles of $F(s + u, \chi)F(s + v, \chi)s^{-1}$. On the three new segments, by Lemmas 2.1, 2.2 and 2.3, we have $|F(s + u, \chi)F(s + v, \chi)| \ll \log^4 T$. Hence the integral of $F(s + u, \chi)F(s + v, \chi)x^s s^{-1}$ over the three segments is

$$\ll (\log^4 x) \left(\int_b^c \frac{x^\sigma}{|\sigma + iT|} d\sigma + \int_{-T}^T \frac{x^b}{|b + it|} dt \right) \ll x^b \log^5 x.$$

This proves (1).

We now consider the case $v \leq \varepsilon$ and $u \geq 2\varepsilon$. We set $b = \frac{1}{2} - \frac{3\varepsilon}{2}$ and deform the contour of integration as in the previous case. Since $u + b \geq \frac{1}{2} + \frac{\varepsilon}{2}$ and $v + b \leq \frac{1}{2} - \frac{\varepsilon}{2}$, we have by Lemma 2.3 that $|F(s + u, \chi)F(s + v, \chi)| \ll \log^4 T \ll \log^4 x$ on all three new segments. As in the proof of (1), the integral over the new contour is $\ll x^b \log^5 x$. We pick up residue terms from poles of $F(s + v, \chi)$ inside the rectangle coming from the nontrivial zeros of $L(s, \chi)$, plus a pole at $s = \frac{1}{2} - v$ from the $\frac{\zeta'}{\zeta}(2s + 2v)$ term if $\chi^2 = \chi_0$. The sum of the residues is

$$\sum_{|\gamma| \leq T} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi)x^{\frac{1}{2}-v+i\gamma}}{\frac{1}{2} - v + i\gamma} - A(\chi) \frac{F(\frac{1}{2} + u - v, \chi)x^{\frac{1}{2}-v}}{1 - 2v},$$

and (2) follows.

Finally, consider the case $0 \leq u, v \leq 2\varepsilon$. Let $b = \frac{1}{2} - 3\varepsilon$ and deform the contour as in the previous cases. As before, the integral over the new contour is $O(x^b \log^5 x)$. This time, we pick up residues from poles of both $F(s + u, \chi)$ and $F(s + v, \chi)$. The sum of the residues is

$$\sum_{|\gamma| \leq T} \left(\frac{F(\frac{1}{2} + u - v + i\gamma, \chi)x^{\frac{1}{2}-v+i\gamma}}{\frac{1}{2} - v + i\gamma} + \frac{F(\frac{1}{2} - u + v + i\gamma, \chi)x^{\frac{1}{2}-u+i\gamma}}{\frac{1}{2} - u + i\gamma} \right) - A(\chi) \left(\frac{F(\frac{1}{2} + u - v, \chi)x^{\frac{1}{2}-v}}{1 - 2v} + \frac{F(\frac{1}{2} - u + v, \chi)x^{\frac{1}{2}-u}}{1 - 2u} \right),$$

and (3) follows. \square

Proof of Lemma 3.1. Begin with

$$\int_0^\infty \int_0^\infty G(x, u, v; \chi) du dv = I_1 + I_2 + 2I_3 + I_4,$$

where I_1 is the integral over $\max(u, v) \geq \log x$, I_2 is the integral over $2\varepsilon \leq \max(u, v) \leq \log x$ and $\min(u, v) \geq \varepsilon$, I_3 is the integral over $0 \leq v \leq \varepsilon$, $2\varepsilon \leq u \leq \log x$, and I_4 is the integral over $0 \leq u, v \leq 2\varepsilon$. For $\max(u, v) \geq \log x$,

$$|G(x, u, v; \chi)| \leq \sum_{p \leq x} \frac{\log p}{p^u} \sum_{q \leq x} \frac{\log q}{q^v} \ll \frac{x}{2^{\max(u, v)}},$$

whence $I_1 \ll x^{1-\log 2}$. By Lemma 4.1 (1), $I_2 \ll x^{1/2-\varepsilon/2} \log^7 x$.

By Lemma 4.1 (2),

$$(4.1) \quad I_3 = x^{1/2} \int_0^\varepsilon \int_{2\varepsilon}^{\log x} \sum_{|\gamma| \leq T} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi) x^{-v+i\gamma}}{\frac{1}{2} - v + i\gamma} - A(\chi) \frac{F(\frac{1}{2} + u - v, \chi) x^{-v}}{1 - 2v} du dv + O(x^{1/2-\frac{3\varepsilon}{2}} \log^6 x).$$

By Lemmas 2.2 and 2.3,

$$(4.2) \quad \int_0^\varepsilon \int_{2\varepsilon}^{\log x} \frac{F(\frac{1}{2} + u - v, \chi) x^{-v}}{1 - 2v} du dv \ll \int_0^\varepsilon x^{-v} dv \ll \frac{1}{\log x}.$$

Let

$$\Sigma_1(x) = (\log x) \int_0^\varepsilon \int_{2\varepsilon}^{\log x} \sum_{0 < |\gamma| < T} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi) x^{-v+i\gamma}}{\frac{1}{2} - v + i\gamma} du dv.$$

Since $\frac{1}{2} + u - v \geq \frac{1}{2} + \varepsilon$ for $0 \leq v \leq \varepsilon$ and $2\varepsilon \leq u \leq \log x$, by Lemmas 2.1, 2.2, and 2.3,

$$F(\frac{1}{2} + u - v + i\gamma, \chi) = -\frac{L'}{L}(\frac{1}{2} + u - v + i\gamma, \chi) + O(1) \ll \log(|\gamma| + 3).$$

We also have $F(1/2 + u - v + i\gamma, \chi) \ll 2^{-u}$ for $u \geq 2$. Thus, for positive integers n ,

$$\int_{2^n}^{2^{n+1}} |\Sigma_1(e^y)|^2 dy \ll 2^{2n} \sum_{|\gamma_1|, |\gamma_2| \leq T} \frac{\log(|\gamma_1| + 3) \log(|\gamma_2| + 3)}{|\gamma_1 \gamma_2|} \times \int_0^\varepsilon \int_0^\varepsilon \left| \int_{2^n}^{2^{n+1}} e^{y(-v_1 + i\gamma_1 - v_2 - i\gamma_2)} dy \right| dv_1 dv_2.$$

The summands with $|\gamma_1 - \gamma_2| < 1$ contribute, by Lemma 2.2,

$$\begin{aligned} &\ll 2^{2n} \sum_{\substack{|\gamma_1|, |\gamma_2| \leq T \\ |\gamma_1 - \gamma_2| < 1}} \frac{\log(|\gamma_1| + 3) \log(|\gamma_2| + 3)}{|\gamma_1| |\gamma_2|} \int_{2^n}^{2^{n+1}} \left(\int_0^\varepsilon e^{-vy} dv \right)^2 dy \\ &\ll 2^n \sum_{|\gamma| \leq T} \frac{\log^3(|\gamma| + 3)}{|\gamma|^2} \ll 2^n. \end{aligned}$$

The summands with $|\gamma_1 - \gamma_2| \geq 1$ contribute, by Lemma 2.4,

$$\ll \sum_{\substack{|\gamma_1|, |\gamma_2| < T \\ |\gamma_1 - \gamma_2| \geq 1}} \frac{2^{2n} \log(|\gamma_1| + 3) \log(|\gamma_2| + 3)}{|\gamma_1| |\gamma_2| |\gamma_1 - \gamma_2|} \left(\int_0^\varepsilon e^{-v2^n} dv \right)^2 \ll 1.$$

Thus, $\int_{2^n}^{2^{n+1}} |\Sigma_1(e^y)|^2 dy = O(2^n)$. Summing over $n \leq \frac{\log Y}{\log 2} + 1$ yields $\int_1^Y |\Sigma_1(e^y)|^2 dy = O(Y)$.

Finally, using Lemma 4.1 (3) gives

$$(4.3) \quad I_4 = x^{1/2} \int_0^{2\varepsilon} \int_0^{2\varepsilon} \sum_{|\gamma| \leq T} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi)x^{-v+i\gamma}}{\frac{1}{2} - v + i\gamma} + \frac{F(\frac{1}{2} - u + v + i\gamma, \chi)x^{-u+i\gamma}}{\frac{1}{2} - u + i\gamma} \\ - A(\chi) \left(\frac{F(\frac{1}{2} + u - v, \chi)x^{-v}}{1 - 2v} + \frac{F(\frac{1}{2} - u + v, \chi)x^{-u}}{1 - 2u} \right) du dv + O(x^{\frac{1}{2}-3\varepsilon} \log^3 x).$$

Now assume $\chi^2 = \chi_0$. We will show that

$$(4.4) \quad - \int_0^{2\varepsilon} \int_0^{2\varepsilon} \frac{F(\frac{1}{2} + u - v, \chi)x^{-v}}{1 - 2v} + \frac{F(\frac{1}{2} - u + v, \chi)x^{-u}}{1 - 2u} du dv = \frac{\log \log x + O(1)}{\log x}.$$

Together with (4.1), (4.2) and (4.3), this completes the proof of Lemma 3.1.

Note that $F(\frac{1}{2} + w) = -\frac{1}{2w} + O(1)$ by Lemmas 2.1 and 2.3. Replacing x with e^y , the left side of (4.4) is

$$= \frac{1}{2} \int_0^{2\varepsilon} \int_0^{2\varepsilon} \frac{e^{-yv}}{(u-v)(1-2v)} + \frac{e^{-yu}}{(v-u)(1-2u)} du dv + O\left(\int_0^{2\varepsilon} \int_0^{2\varepsilon} e^{-yv} du dv\right).$$

The error term above is $O(1/y)$. In the main term, when $|u-v| < 1/y$, the integrand is $O(ye^{-vy})$ and the corresponding part of the double integral is $O(1/y)$. When $u \geq v + 1/y$, the integrand is

$$\frac{e^{-vy}}{u-v} + O\left(\frac{ve^{-vy} + e^{-uy}}{u-v}\right)$$

and the corresponding part of the double integral is

$$\int_0^{2\varepsilon} e^{-vy} \log\left(\frac{y}{2\varepsilon - v}\right) dv + O\left(\frac{1}{y}\right) = \frac{\log y + O(1)}{y}.$$

The contribution from $u \leq v - 1/y$ is, by symmetry, also $\frac{\log y + O(1)}{y}$. The asymptotic (4.4) follows. \square

5. PROOF OF LEMMA 3.2

Lemma 5.1. *Uniformly for $y \geq 1$, $0 < |\xi| \leq 1$, $|w| \geq \frac{1}{2}$ and $a \geq 0$ we have*

$$\left| \int_0^{2\varepsilon} \int_0^{2\varepsilon} \frac{v^a e^{-vy}}{(u-v+i\xi)(w-v)} du dv \right| \ll \frac{(4\varepsilon)^a \log \min(2y, \frac{2}{|\xi|})}{y|w|}.$$

Proof. Let I denote the double integral in the Lemma. If $|\xi| \geq \frac{1}{y}$, then

$$I \ll \frac{1}{|w|} \int_0^{2\varepsilon} v^a e^{-vy} \int_0^{2\varepsilon} \min\left(\frac{1}{|u-v|}, \frac{1}{|\xi|}\right) du dv \\ \ll \frac{(2\varepsilon)^a}{|w|} \left(1 + \log \frac{2}{|\xi|}\right) \int_0^{2\varepsilon} e^{-vy} dv \ll \frac{(2\varepsilon)^a \log(\frac{2}{|\xi|})}{y|w|}.$$

If $|\xi| < \frac{1}{y}$, let $I = I_1 + I_2 + I_3$, where I_1 is the part of I coming from $|u-v| \leq |\xi|$, I_2 is the part of I coming from $|\xi| < |u-v| \leq \frac{1}{y}$, and I_3 is the part of I coming from $|u-v| > \frac{1}{y}$. We have

$$I_1 \ll \frac{1}{|w\xi|} \iint_{\substack{0 \leq u, v \leq 2\varepsilon \\ |u-v| \leq |\xi|}} v^a e^{-vy} du dv \ll \frac{(2\varepsilon)^a}{y|w|}.$$

and

$$I_3 \ll \frac{(2\varepsilon)^a}{|w|} \iint_{\substack{0 \leq u, v \leq 2\varepsilon \\ |u-v| \geq \frac{1}{y}}} \frac{e^{-vy}}{|u-v|} du dv \ll \frac{(2\varepsilon)^a}{|w|} \int_0^{2\varepsilon} e^{-vy} (\log y + 1) dv \ll \frac{(2\varepsilon)^a \log(2y)}{y|w|}.$$

By symmetry,

$$I_2 = \frac{1}{2} \iint_{|\xi| < |u-v| \leq 1/y} \frac{v^a e^{-vy}}{(u-v+i\xi)(w-v)} + \frac{u^a e^{-uy}}{(v-u+i\xi)(w-u)} du dv.$$

Since, $|u^a - v^a| \leq a|u-v|(2\varepsilon)^{a-1}$,

$$(5.1) \quad \begin{aligned} u^a e^{-uy} - v^a e^{-vy} &= e^{-vy} v^a (e^{(v-u)y} - 1) + e^{-vy} (u^a - v^a) e^{(v-u)y} \\ &\ll e^{-vy} y |u-v| (4\varepsilon)^a. \end{aligned}$$

We deduce that

$$\begin{aligned} I_2 &= \iint_{\substack{0 \leq u, v \leq 2\varepsilon \\ |\xi| < |u-v| \leq 1/y}} \frac{(w-u)(u-v)(u^a e^{-uy} - v^a e^{-vy}) + u^a e^{-uy}(u-v)^2 + O(|\xi w|(2\varepsilon)^a e^{-vy})}{2(u-v+i\xi)(v-u+i\xi)(w-u)(w-v)} du dv \\ &\ll \frac{(4\varepsilon)^a}{|w|} \iint_{\substack{0 \leq u, v \leq 2\varepsilon \\ |\xi| < |u-v| \leq 1/y}} y e^{-vy} + \frac{|\xi| e^{-vy}}{|u-v|^2} du dv \ll \frac{(4\varepsilon)^a}{y|w|}. \end{aligned}$$

□

Proof of Lemma 3.2. Let $y = \log x$. We first note by Lemmas 2.1 and 2.2,

$$F\left(\frac{1}{2} + u - v + i\gamma, \chi\right) = \frac{m(\gamma)}{u-v} + R(\gamma, u-v) + R'(\gamma, u-v),$$

where

$$R(\gamma, w) = \sum_{0 < |\gamma' - \gamma| \leq 1} \frac{1}{w + i(\gamma - \gamma')}, \quad R'(\gamma, u-v) = O(\log(|\gamma| + 3)).$$

Then, the double integral in Lemma 3.2 is

$$= \sum_{i=1}^4 \Sigma_{2,i}(y) + 2 \sum_{\substack{|\gamma| \leq T \\ \gamma \text{ distinct}}} m^2(\gamma) e^{iy\gamma} \left(\frac{1}{2} + i\gamma\right) \int_0^{2\varepsilon-2^{-n}} \frac{e^{-yv}}{\frac{1}{2} - v + i\gamma} \int_{v+2^{-n}}^{2\varepsilon} \frac{du}{(u-v)(\frac{1}{2} - u + i\gamma)} dv,$$

where

$$\Sigma_{2,1}(y) = 2 \int_0^{2\varepsilon} \int_0^{2\varepsilon} \sum_{|\gamma| \leq T} \frac{R(\gamma, u-v) e^{y(-v+i\gamma)}}{\frac{1}{2} - v + i\gamma} du dv,$$

$$\Sigma_{2,2}(y) = 2 \int_0^{2\varepsilon} \int_0^{2\varepsilon} \frac{R'(\gamma, u-v) e^{y(-v+i\gamma)}}{\frac{1}{2} - v + i\gamma} du dv,$$

$$\Sigma_{2,3}(y) = \sum_{\substack{|\gamma| \leq T \\ \gamma \text{ distinct}}} m^2(\gamma) e^{iy\gamma} \left(\frac{1}{2} + i\gamma\right) \iint_{\substack{0 \leq u, v \leq 2\varepsilon \\ |u-v| \leq 2^{-n}}} \frac{e^{-yv} - e^{-uy}}{(u-v)(\frac{1}{2} - v + i\gamma)(\frac{1}{2} - u + i\gamma)} dv du,$$

$$\Sigma_{2,4}(y) = 2 \sum_{\substack{|\gamma| \leq T \\ \gamma \text{ distinct}}} m^2(\gamma) e^{iy\gamma} \left(\frac{1}{2} + i\gamma\right) \int_{2^{-n}}^{2\varepsilon} \int_0^{v-2^{-n}} \frac{e^{-yv}}{(u-v)(\frac{1}{2} - v + i\gamma)(\frac{1}{2} - u + i\gamma)} du dv.$$

We show that $\sum_{j=1}^4 \Sigma_{2,j}(y)$ is small in mean square. Note that for $2^n < y \leq 2^{n+1}$, $T = T(e^y)$ is constant. Also, by Lemma 2.2, we have

$$(5.2) \quad m(\gamma) \ll \log(|\gamma| + 3).$$

First, by Lemmas 2.2 and 2.4,

$$\begin{aligned}
\int_{2^n}^{2^{n+1}} |\Sigma_{2,2}(y)|^2 dy &= 4 \iiint\limits_{[0,2\varepsilon]^4} \sum_{\substack{|\gamma_1| \leq T \\ |\gamma_2| \leq T}} \frac{R'(\gamma_1, u_1 - v_1) \overline{R'(\gamma_2, u_2 - v_2)}}{\left(\frac{1}{2} - v_1 + i\gamma_1\right)\left(\frac{1}{2} - v_2 - i\gamma_2\right)} \\
&\quad \times \int_{2^n}^{2^{n+1}} e^{y(-v_1 - v_2 + i\gamma_1 - i\gamma_2)} dy du_j dv_j \\
(5.3) \quad &\ll \sum_{|\gamma_1 - \gamma_2| > 1} \frac{\log(|\gamma_1| + 3) \log(|\gamma_2| + 3)}{|\gamma_1 \gamma_2| \cdot |\gamma_1 - \gamma_2|} \iiint\limits_{[0,2\varepsilon]^4} e^{-2^n(v_1 + v_2)} du_j dv_j \\
&\quad + \sum_{|\gamma_1 - \gamma_2| \leq 1} \frac{\log(|\gamma_1| + 3) \log(|\gamma_2| + 3)}{|\gamma_1 \gamma_2|} \int_{2^n}^{2^{n+1}} \iiint\limits_{[0,2\varepsilon]^4} e^{-y(v_1 + v_2)} du_j dv_j dy \\
&\ll 2^{-n}.
\end{aligned}$$

For the remaining sums, for brevity we define

$$\rho_1 = \frac{1}{2} + i\gamma_1, \quad \rho_2 = \frac{1}{2} - i\gamma_2.$$

Next,

$$\begin{aligned}
\int_{2^n}^{2^{n+1}} |\Sigma_{2,3}(y)|^2 dy &= \int_{2^n}^{2^{n+1}} \sum_{|\gamma_1|, |\gamma_2| \leq T} m(\gamma_1) m(\gamma_2) e^{iy(\gamma_1 - \gamma_2)} \rho_1 \rho_2 \\
&\quad \times \iiint\limits_{\substack{[0,2\varepsilon]^4 \\ |u_j - v_j| \leq 2^{-n}}} \frac{(e^{-v_1 y} - e^{-u_1 y})(e^{-v_2 y} - e^{-u_2 y})}{\prod_{j=1}^2 (u_j - v_j)(\rho_j - v_j)(\rho_j - u_j)} dv_j dy.
\end{aligned}$$

By (5.1), the integrand in the quadruple integral is $\ll y^2 e^{-uy - u_1 y} |\rho_1 \rho_2|^{-2}$. By Lemma 2.2, for a given γ_1 , there are $\ll \log(|\gamma_1| + 3)$ zeros γ_2 with $|\gamma_1 - \gamma_2| < 1$. Hence, the contribution from terms with $|\gamma_1 - \gamma_2| < 1$ is

$$\ll 2^{-n} \sum_{|\gamma_1 - \gamma_2| < 1} \frac{m(\gamma_1) m(\gamma_2)}{|\rho_1 \rho_2|} \ll 2^{-n} \sum_{\gamma_1} \frac{\log^3(|\gamma_1| + 3)}{|\gamma_1|^2} \ll 2^{-n}.$$

Using integration by parts, we have

$$\int_{2^n}^{2^{n+1}} e^{iy(\gamma_1 - \gamma_2)} (e^{-v_1 y} - e^{-u_2 y})(e^{-v_1 y} - e^{-u_2 y}) dy \ll \frac{2^{3n} |u_1 - v_1| |u_2 - v_2| e^{-2^n(u_1 + u_2)}}{|\gamma_1 - \gamma_2|}$$

uniformly in u_1, v_1, u_2, v_2 . Thus, by (5.2) and Lemma 2.4, the contribution from terms with $|\gamma_1 - \gamma_2| \geq 1$ is

$$\ll 2^{-n} \sum_{|\gamma_1 - \gamma_2| \geq 1} \frac{m(\gamma_1) m(\gamma_2)}{|\rho_1 \rho_2| \cdot |\gamma_1 - \gamma_2|} \ll 2^{-n}.$$

Combining these estimates, we have

$$(5.4) \quad \int_{2^n}^{2^{n+1}} |\Sigma_{2,3}(y)|^2 dy \ll 2^{-n}.$$

In the same manner, we have

$$\int_{2^n}^{2^{n+1}} |\Sigma_{2,4}(y)|^2 dy = \sum_{\substack{|\gamma_1| \leq T \\ |\gamma_2| \leq T}} m(\gamma_1) m(\gamma_2) \rho_1 \rho_2 \int_{2^n}^{2^{n+1}} \iiint\limits_{\substack{[0,2\varepsilon]^4 \\ u_j \leq v_j - 2^{-n}}} \frac{e^{y(-v_1 - v_2 + i(\gamma_1 - \gamma_2))}}{\prod_{j=1}^2 (u_j - v_j)(\rho_j - v_j)(\rho_j - u_j)} du_j dv_j dy.$$

The contribution to the right side from terms with $|\gamma_1 - \gamma_2| < 1$ is

$$\begin{aligned} &\ll \sum_{|\gamma_1 - \gamma_2| < 1} \frac{m(\gamma_1)m(\gamma_2)}{|\gamma_1\gamma_2|} \int_{2^n}^{2^{n+1}} \left(\int_{2^{-n}}^{2^\varepsilon} \int_0^{v-2^{-n}} \frac{e^{-yv}}{(v-u)} du dv \right)^2 \\ &\ll \sum_{\gamma_1} \frac{\log^3(|\gamma_1| + 3)}{|\gamma_1|^2} \int_{2^n}^{2^{n+1}} \left(\int_{1/y}^{\infty} e^{-yv} \log(yv) dv \right)^2 \ll 2^{-n}. \end{aligned}$$

The terms with $|\gamma_1 - \gamma_2| > 1$ contribute

$$\begin{aligned} &\ll \sum_{\substack{|\gamma_1|, |\gamma_2| < T \\ |\gamma_1 - \gamma_2| > 1}} \frac{m(\gamma_1)m(\gamma_2)}{|\gamma_1\gamma_2| \cdot |\gamma_1 - \gamma_2|} \left(\int_{2^{-n}}^{2^\varepsilon} \int_0^{v-2^{-n}} \frac{e^{-2^n v}}{v-u} du dv \right)^2 \\ &\ll \sum_{|\gamma_1 - \gamma_2| > 1} \frac{\log(|\gamma_1| + 3) \log(|\gamma_2| + 3)}{|\gamma_1\gamma_2| \cdot |\gamma_1 - \gamma_2|} \left(\frac{1}{2^n} \right)^2 \ll \frac{1}{2^{2n}}. \end{aligned}$$

Therefore,

$$(5.5) \quad \int_{2^n}^{2^{n+1}} |\Sigma_{2,4}(y)|^2 dy \ll 2^{-n}.$$

Estimating an average of $\Sigma_{2,1}(y)$ is more complicated, since $R(\gamma, w)$ could be very large if $|w|$ is small and there is another γ' very close to γ . We get around the problem by noticing that $R(\gamma, w) + R(\gamma, -w)$ is always small. We first have, by (5.1) and Lemma 2.2,

$$(5.6) \quad \int_{2^n}^{2^{n+1}} |\Sigma_{2,1}(y)|^2 dy \ll \sum_{\gamma_1, \gamma_2} \log^2(|\gamma_1| + 3) \log^2(|\gamma_2| + 3) \max_{\substack{0 < |\gamma_1 - \gamma'_1| \leq 1 \\ 0 < |\gamma_2 - \gamma'_2| \leq 1}} \int_{2^n}^{2^{n+1}} e^{iy(\gamma_1 - \gamma_2)} \\ \times \iiint_{[0, 2^\varepsilon]^4} \frac{e^{y(-v_1 - v_2)}}{(u_1 - v_1 + i\xi_1)(\rho_1 - v_1)(u_2 - v_2 + i\xi_2)(\rho_2 - v_2)} du_j dv_j dy,$$

where $\xi_1 = \gamma_1 - \gamma'_1$ and $\xi_2 = -(\gamma_2 - \gamma'_2)$. Let

$$M(\gamma) = \max_{\substack{|\gamma - \gamma_1| \leq 1 \\ 0 < |\gamma_1 - \gamma'_1| < 1}} \frac{2}{|\gamma_1 - \gamma'_1|}.$$

By Lemmas 2.3 and 5.1, the terms with $|\gamma_1 - \gamma_2| < 1$ contribute

$$\begin{aligned} &\ll \sum_{|\gamma_1 - \gamma_2| < 1} \frac{\log^2(|\gamma_1| + 3) \log^2(|\gamma_2| + 3)}{|\gamma_1\gamma_2|} \int_{2^n}^{2^{n+1}} \frac{1}{y^2} \prod_{j=1}^2 \log \left(\min \left(2y, \frac{2}{|\gamma_j - \gamma'_j|} \right) \right) dy \\ &\ll \frac{1}{2^n} \sum_{\gamma_1} \frac{\log^5(|\gamma_1| + 3)}{|\gamma_1|^2} \log^2(\min(2^{n+2}, M(\gamma))) = o\left(\frac{n^2}{2^n}\right) \quad (n \rightarrow \infty). \end{aligned}$$

Now suppose $|\gamma_1 - \gamma_2| > 1$. With $\gamma_1, \gamma_2, \gamma'_1, \gamma'_2$ all fixed, let $\Delta = \gamma_1 - \gamma_2$. Fixing u_1, v_1, u_2, v_2 , we integrate over y first. The quintuple integral in (5.6) is $J(2^{n+1}) - J(2^n)$, where

$$J(y) = e^{iy\Delta} \iiint_{[0, 2^\varepsilon]^4} \frac{e^{-y(v_1 + v_2)}}{(i\Delta - v_1 - v_2) \prod_{j=1}^2 (u_j - v_j + i\xi_j)(\rho_j - v_j)} du_j dv_j.$$

Using

$$\frac{1}{i\Delta - v_1 - v_2} = \frac{1}{i\Delta} \sum_{k=0}^{\infty} \left(\frac{v_1 + v_2}{i\Delta} \right)^k = \sum_{a,b \geq 0} \binom{a+b}{a} \frac{v_1^a v_2^b}{(i\Delta)^{a+b}},$$

together with Lemma 5.1, yields

$$|J(y)| \ll \frac{\log^2 y}{|\rho_1 \rho_2 \Delta| y^2} \sum_{a,b \geq 0} \binom{a+b}{a} \left(\frac{4\varepsilon}{|\Delta|} \right)^{a+b} \ll \frac{\log^2 y}{|\rho_1 \rho_2 \Delta| y^2}.$$

Therefore, by Lemma 2.4,

$$\sum_{\gamma_1, \gamma_2} \log^2(|\gamma_1| + 3) \log^2(|\gamma_2| + 3) \max_{\substack{0 < |\gamma_1 - \gamma_1'| \leq 1 \\ 0 < |\gamma_2 - \gamma_2'| \leq 1}} |J(2^{n+1}) - J(2^n)| \ll \frac{n^2}{2^{2n}},$$

and hence

$$(5.7) \quad \int_{2^n}^{2^{n+1}} |\Sigma_{2,1}(y)|^2 = o(n^2 2^{-n}).$$

Define

$$\Sigma_2(x; \chi) = (\log x) \sum_{j=1}^4 \Sigma_{2,j}(\log x).$$

By (5.3), (5.4), (5.5) and (5.7),

$$\int_2^Y |\Sigma_2(e^y; \chi)|^2 dy \leq 4 \sum_{j=1}^4 \sum_{n \leq \frac{\log Y}{\log 2} + 1} 2^{2n} \int_{2^n}^{2^{n+1}} |\Sigma_{2,j}(y)|^2 dy = o(Y \log^2 Y) \quad (Y \rightarrow \infty).$$

This completes the proof of Theorem 3.2. □

6. PROOF OF LEMMA 3.3

Put $y = \log x$. For any γ we have

$$\begin{aligned} & \int_0^{2\varepsilon - 2^{-n}} \frac{e^{-yv}}{\frac{1}{2} - v + i\gamma} \int_{v+2^{-n}}^{2\varepsilon} \frac{du}{(u-v)(\frac{1}{2} - u + i\gamma)} dv \\ &= \int_0^{2\varepsilon - 2^{-n}} e^{-yv} \left(\frac{1}{\frac{1}{2} + i\gamma} + O\left(\frac{v}{\frac{1}{4} + \gamma^2}\right) \right) \int_{v+2^{-n}}^{2\varepsilon} \left(\frac{1}{\frac{1}{2} + i\gamma} + O\left(\frac{u}{\frac{1}{4} + \gamma^2}\right) \right) \frac{du}{u-v} dv \\ &= \frac{M + E}{(1/2 + i\gamma)^2}, \end{aligned}$$

where

$$M = \int_0^{2\varepsilon - 2^{-n}} e^{-yv} (\log(2\varepsilon - v) + \log 2^n) dv = \frac{\log y + O(1)}{y}$$

and

$$\begin{aligned} E &\ll \int_0^{2\varepsilon - 2^{-n}} e^{-yv} \int_{v+2^{-n}}^{2\varepsilon} \frac{u}{u-v} du dv \\ &\ll \int_0^{2\varepsilon - 2^{-n}} e^{-yv} (1 + v \log 2^n + v \log(2\varepsilon - v)) dv \ll \frac{1}{y}. \end{aligned}$$

Hence, the zeros with $|\gamma| \leq T_0$ contribute

$$\frac{2 \log \log x}{\log x} \sum_{\substack{|\gamma| \leq T_0 \\ \gamma \text{ distinct}}} \frac{m^2(\gamma)x^{i\gamma}}{1/2 + i\gamma} + O\left(\frac{\log^3 T_0}{\log x}\right).$$

Next, let $\Sigma_3(x; T_0)$ be the sum over zeros with $T_0 < |\gamma| \leq T$. We have

$$(6.1) \quad \int_{2^n}^{2^{n+1}} |\Sigma_3(e^y, T_0)|^2 dy \leq \sum_{T_0 \leq |\gamma_1|, |\gamma_2| \leq T} 2^{2n+2} m(\gamma_1)m(\gamma_2) \left(\frac{1}{2} + i\gamma_1\right) \left(\frac{1}{2} - i\gamma_2\right) \int_{2^n}^{2^{n+1}} e^{yi(\gamma_1 - \gamma_2)} \iiint_{u_j \geq v_j + 2^{-n}} \frac{e^{-yv_1 - yv_2}}{\prod_{j=1}^2 (u_j - v_j)(\frac{1}{2} - v_j + i\gamma_j)(\frac{1}{2} - u_j + i\gamma_j)} du_j dv_j dy.$$

The sum over $|\gamma_1 - \gamma_2| < 1$ on the right side of (6.1) is

$$\begin{aligned} &\ll \sum_{\substack{T_0 \leq |\gamma_1|, |\gamma_2| \leq T \\ |\gamma_1 - \gamma_2| < 1}} \frac{2^{2n} m(\gamma_1)m(\gamma_2)}{|\gamma_1||\gamma_2|} \int_{2^n}^{2^{n+1}} \iiint_{u_j \geq v_j + 2^{-n}} \frac{e^{-yv_1 - yv_2}}{(u_1 - v_1)(u_2 - v_2)} du_j dv_j dy \\ &\ll \sum_{\substack{T_0 \leq |\gamma_1|, |\gamma_2| \leq T \\ |\gamma_1 - \gamma_2| < 1}} \frac{n^2 2^n m(\gamma_1)m(\gamma_2)}{|\gamma_1||\gamma_2|} \ll n^2 2^n \sum_{|\gamma| \geq T_0} \frac{\log^3(|\gamma| + 3)}{|\gamma|} \ll \frac{n^2 2^n \log^5 T_0}{T_0}, \end{aligned}$$

applying Lemma 2.2. The terms where $|\gamma_1 - \gamma_2| > 1$ on the right hand side of (6.1) total

$$\begin{aligned} &\ll \sum_{\substack{T_0 \leq |\gamma_1|, |\gamma_2| \leq T \\ |\gamma_1 - \gamma_2| > 1}} \frac{2^{2n} m(\gamma_1)m(\gamma_2)}{|\gamma_1||\gamma_2||\gamma_1 - \gamma_2|} \iiint_{u_j \geq v_j + 2^{-n}} \frac{e^{-2^n v_1 - 2^n v_2}}{(u_1 - v_1)(u_2 - v_2)} du_j dv_j \\ &\ll \sum_{\substack{T_0 \leq |\gamma_1|, |\gamma_2| \\ |\gamma_1 - \gamma_2| > 1}} \frac{n^2 \log(|\gamma_1| + 3) \log(|\gamma_2| + 3)}{|\gamma_1||\gamma_2||\gamma_1 - \gamma_2|} \ll n^2 \frac{\log^5 T_0}{T_0}. \end{aligned}$$

by Lemma 2.4. Summing over n proves the lemma.

REFERENCES

- [Ch] P. L. Chebyshev, *Lettre de M. le professeur Tchébychev à M. Fuss, sur un nouveau théoreme relatif aux nombres premiers contenus dans la formes $4n + 1$ et $4n + 3$* , Bull. de la Classe phys.-math. de l'Acad. Imp. des Sciences St. Petersburg **11** (1853), 208.
- [Da] H. Davenport, *Multiplicative Number Theory, 3rd ed., Graduate Texts in Mathematics vol. 74*, Springer-Verlag, New York-Berlin, 2000.
- [FK] K. Ford and S. Konyagin, *Chebyshev's conjecture and the prime number race*. IV International Conference "Modern Problems of Number Theory and its Applications": Current Problems, Part II (Russian) (Tula, 2001), 67–91, Mosk. Gos. Univ. im. Lomonosova, Mekh.-Mat. Fak., Moscow, 2002.
- [GM] A. Granville and G. Martin, *Prime number races*. Amer. Math. Monthly **113** (2006), no. 1, 1–33.
- [KT] S. Knapowski and P. Turán, *Comparative Prime Number Theory I*, Acta. Math. Sci. Hungar. **13** (1962), 315-342.
- [La] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen, 3rd ed.*, Chelsea, New York, 1974.
- [Le] J. Leech, *Note on the distribution of prime numbers*, J. London Math. Soc. **32** (1957), 56-58.
- [Li] J. E. Littlewood, *Sur la distribution des nombres premiers*, C. R. Acad. des Sciences Paris **158** (1914), 1869-1872.
- [RS] M. Rubinstein and P. Sarnak, *Chebyshev's Bias*, J. Exper. Math. **3** (1994), 173-197.
- [S] J. Sneed, *Lead changes in the prime number race*, submitted.

E-mail address: ford@math.uiuc.edu, jpsneed@uiuc.edu