Problem 1. Show that a path-connected space is weakly equivalent to a product of Eilenberg-MacLane spaces if and only if it admits a Postnikov tower of principal fibrations with trivial $k$-invariants (all of them).

Note. Here, we follow Hatcher’s convention that the $k$-invariants are used to build the Postnikov tower of $X$ starting from $P_1X$ and not $P_0X$. In other words, by “Postnikov tower of principal fibrations”, we mean that the maps $P_nX \to P_{n-1}X$ are principal fibrations for all $n \geq 2$. Using $n \geq 1$ instead would force $\pi_1X$ to be abelian.

Solution. Some preliminary observations.

1. A path-connected space $X$ is weakly equivalent to a product of Eilenberg-MacLane spaces if and only if it is weakly equivalent to $\prod_{i \geq 1} K(\pi_iX, i)$.

2. A projection $\pi_B: B \times F \to B$ is always a fibration, so that the sequence

$$F \xrightarrow{\iota} B \times F \xrightarrow{\pi_B} B$$

is a fiber sequence. Here $\iota = (c_{b_0}, \text{id}_F): F \to B \times F$ denotes the “slice inclusion” $\iota(f) = (b_0, f)$.

3. The homotopy fiber of the constant map $c: X \to Y$ is

$$F(c) = \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = c(x), \gamma(1) = y_0\}$$

$$= \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = \gamma(1) = y_0\}$$

$$= X \times \Omega Y.$$  

Iterating the homotopy fiber once more yields the fiber sequence

$$\Omega Y \xrightarrow{\iota} X \times \Omega Y \xrightarrow{\pi_X} X \xrightarrow{c} Y.$$  

In particular, the fibration $\pi_B: B \times F \to B$ can be extended to the right by the constant map if and only if $F$ admits a delooping.

Now onto the proof of the statement.

($\Rightarrow$) Assume given a (zigzag of, but WLOG a single) weak equivalence $\varphi: X \xrightarrow{\sim} \prod_{i \geq 1} K(\pi_iX, i)$.
Then the successive projections

\[ \vdots \]

\[ \prod_{i=1}^{n} K(\pi_i X, i) = P_n X \]

\[ \vdots \]

\[ \prod_{i=1}^{n-1} K(\pi_i X, i) = P_{n-1} X \]

\[ \vdots \]

\[ K(\pi_1 X, 1) \times K(\pi_2 X, 2) = P_2 X \]

\[ X \xrightarrow{\phi} \prod_{i \geq 1} K(\pi_i X, i) \]

\[ K(\pi_1 X, 1) = P_1 X \]

\[ * = P_0 X \]

form a Postnikov tower for \( X \).

The truncation map between successive stages \( P_n X \to P_{n-1} X \) is a projection with fiber \( K(\pi_n X, n) \), which admits a delooping \( K(\pi_n X, n+1) \) since \( \pi_n X \) is abelian (as \( n \geq 2 \)).

By observation (3), \( P_n X \to P_{n-1} X \) is a principal fibration which can be extended to the right by the constant map

\[ P_n X \to P_{n-1} X \xrightarrow{\ast} K(\pi_n X, n+1) \]

so that the \( k \)-invariant \( k_{n-1} \in H^{n+1}(P_{n-1} X; \pi_n X) \) is trivial.
(⇐) Assume all $k$-invariants of $X$ are trivial, i.e. for all $n \geq 2$ we have fiber sequences

$$P_nX \to P_{n-1}X \xrightarrow{\sim} K(\pi_nX, n + 1).$$

By observation (3), this implies the equivalence

$$P_nX \simeq P_{n-1}X \times \Omega K(\pi_nX, n + 1) \simeq P_{n-1}X \times K(\pi_nX, n).$$

Repeating this equivalence inductively, we conclude that for all $n \geq 1$, the Postnikov stages of $X$ are products

$$P_nX \simeq \prod_{i=1}^{n} K(\pi_iX, i).$$

Since these truncation maps $P_nX \to P_{n-1}X$ are projections, in particular fibrations, the homotopy limit of the tower is (equivalent to) its strict limit. We conclude:

$$X \xrightarrow{\sim} \text{holim}_n P_nX$$

$$\simeq \text{holim}_n \prod_{i=1}^{n} K(\pi_iX, i)$$

$$\simeq \lim_n \prod_{i=1}^{n} K(\pi_iX, i)$$

$$\simeq \prod_{i=1}^{\infty} K(\pi_iX, i)$$

and thus $X$ is weakly equivalent to a product of Eilenberg-MacLane spaces. □
Problem 2. Let $X$ be a path-connected CW complex and $G$ a group. Show that the map
\[ \pi_1: [X, K(G, 1)]_* \to \text{Hom}_{\text{gp}}(\pi_1(X), G) \]
is a bijection.

Solution. WLOG $X$ has a single 0-cell. Indeed, $X$ is pointed homotopy equivalent to such a CW complex, and the functors on both sides $[-, K(G, 1)]_*$ and $\text{Hom}_{\text{gp}}(\pi_1(-), G)$ are invariant under pointed homotopy equivalence.

WLOG $X$ is 2-dimensional. Indeed, the skeletal inclusion $\iota_2: X_2 \hookrightarrow X$ induces an isomorphism $\iota_2^*: \pi_1(X_2) \cong \pi_1(X)$ and a bijection
\[ \iota_2^*: [X, K(G, 1)]_* \cong [X_2, K(G, 1)]_* \]
as shown in the notes from 5/29.

True for wedges of circles. When $X$ is a wedge of circles $X \simeq \bigvee_{j \in J} S^1$, then $\pi_1$ does induce a bijection, as shown by the commutative diagram:

\[
\begin{array}{ccc}
[V_j S^1, K(G, 1)]_* & \xrightarrow{\pi_1} & \text{Hom}_{\text{gp}}(\pi_1(V_j S^1), G) \\
\cong & & \cong \\
\Pi_j [S^1, K(G, 1)]_* & \xrightarrow{\Pi_j \pi_1} & \Pi_j \text{Hom}_{\text{gp}}(\pi_1(S^1), G) \\
\cong & & \cong \\
\Pi_j \pi_1 K(G, 1) & \xrightarrow{\Pi_j \psi} & \Pi_j G
\end{array}
\]

where $\psi: \pi_1 K(G, 1) \cong G$ is some fixed identification.

True in general. Let $X$ be a 2-dimensional CW complex with a single 0-cell. WLOG all attaching maps of the 2-cells are pointed, so that $X = X_2$ sits in a cofiber sequence
\[ \bigvee S^1 \to X_1 \to X_2 \to \bigvee S^2. \] (1)

By the theorem on the fundamental group of CW complexes, applying $\pi_1$ to this specific cofiber sequence (1) yields a right exact sequence of groups
\[ \pi_1(\bigvee S^1) \to \pi_1(X_1) \to \pi_1(X_2) \to 0. \]

Applying $\text{Hom}_{\text{gp}}(-, G)$ then yields a left exact sequence of pointed sets, which is the bottom row in the diagram below.
Applying $[-, \text{K}(G, 1)]_*$ to the cofiber sequence (1) yields an exact sequence of pointed sets. The natural transformation $\pi_1$ yields a map of exact sequences:

$$\begin{align*}
\left[\bigvee S^2, \text{K}(G, 1)\right]_* = 0 & \quad \longrightarrow & \quad \left[\bigvee S^1, \text{K}(G, 1)\right]_* & \quad \longrightarrow & \quad \left[\bigvee S^1, \text{K}(G, 1)\right]_* \\
0 & \quad \longrightarrow & \quad \text{Hom}_{\text{Gp}} (\pi_1(X_2), G) & \quad \longrightarrow & \quad \text{Hom}_{\text{Gp}} (\pi_1(X_1), G) & \quad \longrightarrow & \quad \text{Hom}_{\text{Gp}} (\pi_1(\bigvee S^1), G).
\end{align*}$$

Because $X_1 \simeq \bigvee S^1$ is also a wedge of circles, the two downward maps to the right are bijections, and hence so is the downward map

$$\pi_1: \left[\bigvee S^2, \text{K}(G, 1)\right]_* \rightarrow \text{Hom}_{\text{Gp}} (\pi_1(X_2), G). \quad \square$$
Problem 3. Let $X$ be a CW complex, with $n$-skeleton $X_n$, and let $Y$ be a path-connected simple space. Let $n \geq 2$, and let $f_n, g_n : X_n \to Y$ be two maps which agree on $X_{n-1}$, i.e.

$$f_n|_{X_{n-1}} = g_n|_{X_{n-1}}.$$ Let $d(f_n, g_n) \in C^n(X; \pi_n Y)$ denote their difference cochain.

Show that $f_n \simeq g_n \text{ rel } X_{n-2}$ holds if and only if $[d(f_n, g_n)] = 0 \in H^n(X; \pi_n Y)$ holds, i.e. $d(f_n, g_n)$ is a coboundary.

Solution. Since $Y$ is path-connected and simple, we can safely ignore basepoints and work with unpointed maps.

WLOG $X = X_n$.

Consider the map

$$S : (X_n \times \partial I) \cup (X_{n-1} \times I) \to Y$$

defined by

$$S|_{X_n \times \{0\}} = f_n$$
$$S|_{X_n \times \{1\}} = g_n$$
$$S|_{X_{n-1} \times \{t\}} = f_{n-1} \text{ for all } t \in I.$$

(The letter $S$ was chosen for “Stationary”.)

The condition $f_n \simeq g_n \text{ rel } X_{n-2}$ can be stated as being able to extend the restriction

$$S_{n-1} := S|_{X_n \times \partial I \cup X_{n-2} \times I}$$

to all of $X_n \times I$. In other words, $S = S_n$ is defined on the relative $n$-skeleton of the relative CW complex

$$(X_n \times I, X_n \times \partial I)$$

and we want to extend its restriction $S_{n-1}$ from the relative $(n-1)$-skeleton to the relative $(n+1)$-skeleton $X_n \times I$. There exists such an extension if and only if the obstruction class of $S_n$

$$c(S_n) \in C^{n+1}(X_n \times I, X_n \times \partial I; \pi_n Y)$$

is a coboundary.

The short exact sequence of cellular chain complexes

$$0 \to C_*(X \times \partial I) \to C_*(X \times I) \to C_*(X \times I, X \times \partial I) \to 0$$

yields a short exact sequence of cellular cochain complexes

$$0 \to C^*(X \times I, X \times \partial I; \pi_n Y) \to C^*(X \times I; \pi_n Y) \to C^*(X \times \partial I; \pi_n Y) \to 0.$$ Using the fact that $C_*(I)$ is finitely generated and free in each degree, we obtain the isomorphism

$$C^{n+1}(X_n \times I, X_n \times \partial I; \pi_n Y) \cong C^n(X_n; \pi_n Y) \otimes C^1(I) \quad (2)$$
and moreover, the coboundary operator in the relative cellular cochain complex $C^\ast(X_n \times I, X_n \times \partial I; \pi_n Y)$ corresponds to the coboundary in $C^\ast(X_n; \pi_n Y)$. In other words, the diagram

$$
\begin{array}{ccc}
C^{n+1}(X_n \times I, X_n \times \partial I; \pi_n Y) & \cong & C^n(X_n; \pi_n Y) \otimes \mathbb{Z} C^1(I) \\
\delta & & \delta \otimes \text{id} \\
C^n(X_n \times I, X_n \times \partial I; \pi_n Y) & \cong & C^{n-1}(X_n; \pi_n Y) \otimes \mathbb{Z} C^1(I) \\
\end{array}
$$

commutes.

Therefore, the obstruction class $c(S_n) \in C^{n+1}(X_n \times I, X_n \times \partial I; \pi_n Y)$ is a coboundary if and only if the corresponding cochain in $C^n(X_n; \pi_n Y)$ is a coboundary.

Relative $(n+1)$-cells of $(X_n \times I, X_n \times \partial I)$ are of the form $e_n^\alpha \times e_1$ for some $n$-cell $e_n^\alpha$ of $X_n$ with attaching map $\varphi_\alpha : S^{n-1} \to X_{n-1}$ and characteristic map

$$
\Phi_\alpha : (D^n, S^{n-1}) \to (X_n, X_{n-1}).
$$

Here $e_1$ denotes the unique 1-cell of the interval $I$.

The value of the cochain $c(S_n)$ on the relative $(n+1)$-cell $e_n^\alpha \times e_1$ is the composite

$$
\partial(D^n \times D^1) \longrightarrow (X_n \times I)_n \longrightarrow Y.
$$

By definition of $S$, that composite is homotopic to the map $d(f_n, g_n)(e_n^\alpha \times e_1) \in \pi_n Y$ (or minus it, depending on our sign convention in the definition of the difference construction). This proves the equality

$$
c(S_n) = \pm d(f_n, g_n)
$$

via the isomorphism (2).

Therefore the obstruction class $c(S_n)$ is a coboundary if and only if the difference cochain $d(f_n, g_n)$ is a coboundary.