Problem 1. Consider the standard inclusions $\mathbb{C}^0 \to \mathbb{C}^1 \to \ldots \to \mathbb{C}^n \to \mathbb{C}^{n+1} \to \ldots$ given by appending a zero in the last coordinate:

$$
\begin{bmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_n 
\end{bmatrix} \mapsto
\begin{bmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_n \\
  0
\end{bmatrix}.
$$

These give rise to inclusions $\ldots \to U(n) \to U(n+1) \to \ldots$ described in terms of matrices by:

$$M \mapsto \begin{bmatrix}
  M & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 1
\end{bmatrix},$$

where $U(n)$ denotes the Lie group of $n \times n$ unitary matrices with complex coefficients.

a. Show that the connectivity of the map $U(n) \to U(n+1)$ goes to infinity as $n$ goes to infinity.

Solution. For each $n \geq 1$, consider the evaluation map

$$p: U(n) \to S^{2n-1}
M \mapsto M(e_n)$$

which picks out the last column of the matrix $M$, viewed as a unit vector in $\mathbb{C}^n$. Here $\{e_1, e_2, \ldots, e_n\}$ denotes the standard basis of $\mathbb{C}^n$.

The map $p$ is clearly surjective, and is moreover a fibration (in fact a fiber bundle), with strict fiber $U(n-1)$, yielding the fiber sequence:

$$U(n-1) \hookrightarrow U(n) \xrightarrow{p} S^{2n-1}.$$

The homotopy fiber $\Omega S^{2n-1}$ of the inclusion $U(n-1) \hookrightarrow U(n)$ is $(2n-3)$-connected, so that the inclusion $U(n-1) \hookrightarrow U(n)$ is $(2n-2)$-connected. $\square$
b. Denote the infinite union $U := \colim_n U(n)$. Show that its homotopy groups satisfy

$$\pi_k U \cong \colim_n \pi_k U(n)$$

and using part (a), find $n$ large enough (as a function of $k$) to guarantee that the map $U(n) \to U$ induces an isomorphism $\pi_k U(n) \cong \pi_k U$.

**Solution.** Since each inclusion $U(n-1) \hookrightarrow U(n)$ is a closed embedding, Corollary 2.5.6 of May-Ponto applies, providing the desired isomorphism

$$\pi_k U \cong \colim_n \pi_k U(n).$$

Note that $U(1) = S^1$ is path-connected, and thus so are all subsequent $U(n)$ for $n \geq 1$.

The connectivity estimate of part (a) guarantees the following. For $k < 2n$, not only is $\pi_k U(n) \cong \pi_k U(n+1)$ an isomorphism, but so are all subsequent induced maps on $\pi_k$:

$$\pi_k U(n) \cong \pi_k U(n+1) \cong \pi_k U(n+2) \cong \ldots$$

which proves the isomorphism $\pi_k U(n) \cong \colim_m \pi_k U(m) \cong \pi_k U$.

Therefore the following condition guarantees that $n$ is large enough:

$$k \leq 2n - 1 \iff k + 1 \leq 2n$$

$$\iff \frac{k + 1}{2} \leq n$$

$$\iff n \geq \left\lceil \frac{k + 1}{2} \right\rceil.$$ 

For example, the low-dimensional homotopy groups $\pi_k U$ are achieved at the following stages:

$$\pi_0 U \cong \pi_0 U(1) = *$$

$$\pi_1 U \cong \pi_1 U(1)$$

$$\pi_2 U \cong \pi_2 U(2)$$

$$\pi_3 U \cong \pi_3 U(2)$$

$$\pi_4 U \cong \pi_4 U(3)$$

$$\pi_5 U \cong \pi_5 U(3)$$

etc. 

$\square$
c. Compute $\pi_k U$ for $0 \leq k \leq 3$.

**Solution.** From part (b), we already know $\pi_0 U = *$ and $\pi_1 U \cong \pi_1 U(1) = \pi_1 S^1 \cong \mathbb{Z}$.

Since the inclusion $U(1) \hookrightarrow U(2)$ is 2-connected, it induces a surjection $0 = \pi_2(U(1)) \twoheadrightarrow \pi_2(U(2))$ which proves $\pi_2(U(2)) = 0$. From part (b), we obtain $\pi_2 U \cong \pi_2 U(2) = 0$.

In fact, we can extract more information from the fiber sequence $U(1) \hookrightarrow U(2) \twoheadrightarrow S^3$. The long exact sequence on homotopy

$$\ldots \rightarrow \pi_k S^1 \rightarrow \pi_k U(2) \rightarrow \pi_k S^3 \rightarrow \pi_{k-1} S^1 \rightarrow \ldots$$

provides the isomorphism $\pi_k U(2) \cong \pi_k S^3$ for all $k \geq 3$. In particular, we obtain $\pi_3 U(2) \cong \pi_3 S^3 \cong \mathbb{Z}$. From part (b), we obtain $\pi_3 U \cong \pi_3 U(2) \cong \mathbb{Z}$.

In summary, the first few homotopy groups of $U$ are:

- $\pi_0 U = *$
- $\pi_1 U = \mathbb{Z}$
- $\pi_2 U = 0$
- $\pi_3 U = \mathbb{Z}$. $\square$
Problem 2. Let \((X, e)\) be a pointed space. The **James construction** on \(X\) is the pointed space obtained by taking words in the elements of \(X\) and declaring that \(e\) is a unit. Formally, it is the quotient space:

\[
J(X) := \bigsqcup_{k \geq 1} X^k/\sim
\]

where \(\sim\) is the equivalence relation generated by identifications of the form:

\[
(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_k) \sim (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k).
\]

a. Show that \(J(X)\) is a topological monoid (under concatenation of words).

**Solution.** \(J(X)\) under concatenation of words is the free monoid on the underlying pointed set of \((X, e)\). It remains to check that it is a topological monoid, i.e. that the multiplication map

\[
\mu: J(X) \times J(X) \to J(X)
\]

is continuous. Note that the unit map \(* \to J(X)\) is automatically continuous.

The multiplication map on \(J(X)\) is induced from the multiplication on the free semigroup \(\bigsqcup_{k \geq 1} X^k\) on \(X\), i.e. before declaring that \(e \in X\) is a unit. This is illustrated in the commutative diagram

\[
\begin{tikzcd}
(\bigsqcup_{n \geq 1} X^n) \times (\bigsqcup_{m \geq 1} X^m) \arrow[r, \mu] \arrow[d, q \times q] & (\bigsqcup_{k \geq 1} X^k) \arrow[d, q]
\arrow[r, \mu] & J(X) \times J(X) \arrow[r, \mu] & J(X)
\end{tikzcd}
\]

where \(q: \bigsqcup_{k \geq 1} X^k \to J(X)\) denotes the quotient map.

In convenient \textbf{Top} (say, compactly generated weakly Hausdorff spaces), a product of two quotient maps is still a quotient map, so that \(q \times q\) is a quotient map. Therefore, to prove continuity of the bottom map \(\mu\), it suffices to prove continuity of the top map \(\mu\).

In naive \textbf{Top} as well as in convenient \textbf{Top}, the functor \(X \times -\) preserves arbitrary coproducts. Therefore, the top map \(\mu\) is naturally isomorphic to the top map in the commutative diagram:

\[
\begin{tikzcd}
\bigsqcup_{n,m \geq 1} X^n \times X^m \arrow[r, \mu] \arrow[d, \mu_{n,m}] & \bigsqcup_{k \geq 1} X^k \arrow[u, \mu_{n,m}]
\arrow[r, \mu] & \bigsqcup_{n,m \geq 1} X^{n+m}
\end{tikzcd}
\]

The map \(\bigsqcup_{n,m} \mu_{n,m}\) is continuous (in fact a homeomorphism) since each \(\mu_{n,m} X^n \times X^m \to X^{n+m}\) is continuous (in fact a homeomorphism). The upward map

\[
\bigsqcup_{n,m \geq 1} X^{n+m} \to \bigsqcup_{k \geq 1} X^k
\]

is continuous, since its restriction to any summand \(X^{n+m} \to \bigsqcup_{k \geq 1} X^k\) is continuous (being just a summand inclusion). \(\square\)
b. Let $M$ be a topological monoid and $f: X \to M$ a pointed map. Show that there is a unique continuous map of monoids $\tilde{f}: J(X) \to M$ making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & M \\
\downarrow{\iota_1} & & \downarrow{\tilde{f}} \\
J(X) \\
\end{array}
$$

commute. Here $\iota_1: X \to J(X)$ denotes the canonical “inclusion of single-letter words”, i.e. the composite

$$
X = X^1 \hookrightarrow \coprod_{k \geq 1} X^k \to J(X).
$$

**Solution.** Since $J(X)$ is the free monoid on the underlying pointed set $(X,e)$, there is a unique map of monoids $\tilde{f}: J(X) \to M$ making the diagram commute. Explicitly, it is given by

$$
\tilde{f}(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n)
$$

which is indeed well defined since $f$ is pointed, that is, $f(e) = 1_M$.

It remains to show that $\tilde{f}$ is continuous. Since $J(X)$ has a quotient topology, it suffices to show that the composite

$$
\begin{array}{ccc}
\coprod_{n \geq 1} X^n & \xrightarrow{q} & J(X) \\
\downarrow{q} & & \downarrow{\tilde{f}} \\
J(X) & \xrightarrow{\tilde{f} \circ q} & M \\
\end{array}
$$

is continuous. But restricted to each summand $X^n$, the composite $\tilde{f} \circ q|_{X^n}: X^n \to M$ is the map given by

$$
\tilde{f} \circ q(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n)
$$

which is the composite

$$
\begin{array}{ccc}
X^n & \xrightarrow{f^n} & M^n \\
& & \xrightarrow{\mu_n} M \\
\end{array}
$$

of two continuous maps. Here $\mu_n: M^n \to M$ is the multiplication map of $n$ inputs, which is unambiguously defined since $M$ is strictly associative, and moreover $\mu_n$ is continuous.

**Upshot.** This shows that $J(X)$ is in fact the free topological monoid on $X$. In other words, let $U: \textbf{TopMon} \to \textbf{Top}_*$ denote the forgetful functor from topological monoids to pointed spaces. Then the functor $J: \textbf{Top}_* \to \textbf{TopMon}$ is left adjoint to $U$, and $\iota_1: X \to J(X)$ is the unit map of the adjunction.
**Definition.** Let \((X,x_0)\) be a pointed space. The space of **Moore loops** \(\Omega_M X\) in \(X\) is the space of pairs \((\gamma, \tau)\) with \(\tau \in [0, \infty)\) and \(\gamma: [0, \tau] \to X\) a loop at the basepoint, i.e. a continuous map satisfying \(\gamma(0) = \gamma(\tau) = x_0\). It is topologized as the subspace:

\[
\Omega_M X = \{ (\gamma, \tau) \in \text{Map} ([0, \infty), X) \times [0, \infty) \mid \gamma(0) = x_0 \text{ and } \gamma(t) = x_0 \text{ for all } t \geq \tau \}
\subseteq \text{Map} ([0, \infty), X) \times [0, \infty).
\]

The basepoint of \(\Omega_M X\) is the “instantaneous loop” \(c_0 := (\gamma, 0)\).

Concatenation of Moore loops is defined as follows: \((\gamma_1, \tau_1) \ast (\gamma_2, \tau_2) \in \Omega_M X\) is the Moore loop \((\gamma, \tau_1 + \tau_2)\) given by

\[
\gamma(t) = \begin{cases} 
\gamma_1(t) & \text{if } 0 \leq t \leq \tau_1 \\
\gamma_2(t - \tau_1) & \text{if } \tau_1 \leq t \leq \tau_1 + \tau_2
\end{cases}
\]

also denoted \(\gamma = \gamma_1 \ast_M \gamma_2\) by abuse of notation.

Concatenation makes \(\Omega_M X\) into a (strict) monoid with unit \(c_0\), and moreover one can check that it is a topological monoid, i.e. the concatenation map

\[
\ast: \Omega_M X \times \Omega_M X \to \Omega_M X
\]

is continuous.
Problem 3.

a. Show that the usual loop space $\Omega X$ and the Moore loop space $\Omega_M X$ are naturally homotopy equivalent, by an equivalence $\varphi: \Omega X \cong \Omega_M X$ which is moreover an $H$-map, i.e. such that the diagram

$$
\begin{array}{ccc}
\Omega X \times \Omega X & \xrightarrow{\varphi \times \varphi} & \Omega_M X \times \Omega_M X \\
\text{concatenation} & & \text{concatenation} \\
\Omega X & \xrightarrow{\varphi} & \Omega_M X
\end{array}
$$

commutes up to homotopy.

Solution. Define $\varphi: \Omega X \to \Omega_M X$ by

$$
\varphi(\gamma) = (\gamma, 1)
$$

where $\gamma: [0, 1] \to X$ is a loop in $X$ based at $x_0$, with standard parametrization by the unit interval. Clearly $\varphi$ is continuous, and is natural in $X$.

Note that $\varphi$ is not a pointed map, but it does send the basepoint to the basepoint component.

Define $\psi: \Omega_M X \to \Omega X$ by

$$
\psi(\gamma, \tau) = \gamma_{\tau}
$$

where the latter denotes the loop $\gamma_{\tau}: [0, 1] \to X$ rescaled by a factor of $\tau$:

$$
\gamma_{\tau}(t) := \gamma(\tau t).
$$

Clearly $\psi$ is continuous, and is natural in $X$.

One composite is the identity, namely $\psi \varphi = \text{id}_{\Omega X}: \Omega X \to \Omega X$.

The other composite $\varphi \psi: \Omega_M X \to \Omega_M X$ is homotopic to the identity, via the homotopy

$$
H(\gamma, \tau, s) = \left(\gamma((1-s)+s\tau), \frac{\tau}{(1-s)+s\tau}\right)
$$

for $s \in [0, 1]$. Indeed, $H$ is continuous and satisfies

$$
H(\gamma, \tau, 0) = (\gamma_1, \tau) = (\gamma, \tau)
$$

$$
H(\gamma, \tau, 1) = (\gamma_\tau, 1) = \varphi \psi(\gamma, \tau).
$$

It remains to show that $\varphi: \Omega X \to \Omega_M X$ preserves concatenation up to homotopy. Let $\alpha, \beta \in \Omega X$. The two ways of going around the diagram (1) yield:

$$
\varphi(\alpha \ast \beta) = (\alpha \ast \beta, 1)
$$

$$
\varphi(\alpha) \ast \varphi(\beta) = (\alpha, 1) \ast (\beta, 1)
$$

$$
= (\alpha \ast_{M} \beta, 2).
$$
Consider the homotopy $G : \Omega X \times \Omega X \times I \to \Omega M X$ given by
\[ G(\alpha, \beta, s) = \left( \frac{\alpha_{1+s}}{1+s}, \frac{1}{1+s} \right) * \left( \frac{\beta_{1+s}}{1+s}, \frac{1}{1+s} \right). \]

Then $G$ is indeed continuous, and it satisfies
\[ G(\alpha, \beta, 0) = (\alpha_1, 1) * (\beta_1, 1) = \varphi(\alpha) * \varphi(\beta) \]
\[ G(\alpha, \beta, 1) = \left( \alpha_2, \frac{1}{2} \right) * \left( \beta_2, \frac{1}{2} \right) = (\alpha * \beta, 1) = \varphi(\alpha * \beta). \]
b. Deduce that the canonical map $\eta: X \to \Omega \Sigma X$ naturally extends up to homotopy to an $H$-map $\tilde{\eta}: J(X) \to \Omega \Sigma X$. Here $J(X)$ denotes the James construction on $X$ (c.f. Problem 2). “Extension up to homotopy” means that $\tilde{\eta}$ makes the following diagram commute up to homotopy:

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & \Omega \Sigma X \\
\downarrow{\iota_1} & & \searrow{\tilde{\eta}} \\
J(X) & & \\
\end{array}
\]

Solution. Consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & \Omega \Sigma X & \xrightarrow{\varphi} & \Omega_M \Sigma X \\
\downarrow{\iota_1} & & \searrow{\psi} & & \nearrow{\eta'} \\
J(X) & & \end{array}
\]

Since $\Omega_M \Sigma X$ is a (strict) topological monoid, there is a unique continuous map of monoids $\eta': J(X) \to \Omega_M \Sigma X$ satisfying $\eta' \circ \iota_1 = \varphi \circ \eta$.

Take $\tilde{\eta} := \psi \circ \eta'$. Since $\psi: \Omega_M X \xrightarrow{\simeq} \Omega X$ is a homotopy equivalence, the left triangle commutes up to homotopy:

\[
\tilde{\eta} \circ \iota_1 = \psi \circ \eta' \circ \iota_1 \\
= \psi \circ \varphi \circ \eta \\
\simeq \text{id}_{\Omega \Sigma X} \circ \eta \\
= \eta.
\]

Since $\eta'$ is a map of monoids, it is in particular an $H$-map. Since $\psi$ is also an $H$-map, the composite $\tilde{\eta} = \psi \circ \eta'$ is also an $H$-map.

Since $\varphi$ and $\psi$ are both natural in $X$, then so is $\eta': J(X) \to \Omega \Sigma X$. 

$\square$