Problem 1. Find a space $X$ such that no choice of basepoint will make it well-pointed.

Solution. Consider the space $\mathbb{Q}$ of rational numbers, with its standard metric topology. For any point $q \in \mathbb{Q}$, there is a homeomorphism $\mathbb{Q} \xrightarrow{\cong} \mathbb{Q}$ sending $q$ to 0, e.g. translation by $-q$. Therefore it suffices to show that the inclusion of some basepoint, say, $i: \{7\} \hookrightarrow \mathbb{Q}$ is not a cofibration.

Consider the mapping cylinder $M(i) \cong \mathbb{Q} \times \{0\} \cup \{7\} \times I \subset \mathbb{Q} \times I$, with the natural maps $f: \mathbb{Q} \rightarrow M(i)$ and $H: \{7\} \times I \rightarrow M(i)$. Recall that $\mathbb{Q}$ is totally disconnected (i.e. connected components are all singletons), and the same argument shows that singletons $\{(q,0)\} \subset M(i)$ with $q \neq 7$ are connected components of $M(i)$. In particular, any path in $M(i)$ starting at $(q,0)$ must be constant. Therefore, any homotopy $G$ from $f$ to a map $g: \mathbb{Q} \rightarrow M(i)$ must satisfy

$$G(q,t) = G(q,0) = f(q) = (q,0)$$

for all $t \in I$ and all $q \neq 7$. At time $t = 1$, the resulting map $g: \mathbb{Q} \rightarrow M(i)$ satisfies $g(q) = (q,0)$ for all $q \neq 7$. By continuity, it also satisfies

$$g(7) = (7,0) \neq (7,1) = H(7,1).$$

Therefore the homotopy $H: \{7\} \times I \rightarrow M(i)$ cannot be extended to a homotopy $\tilde{H}: \mathbb{Q} \times I \rightarrow M(i)$.

$\square$
**Problem 2.** (May § 9.4 Lemma) Show that for all $n \geq 0$, the functor $\pi_n: \text{Top}_* \to \text{Set}_*$ preserves products. In other words, for all pointed spaces $X$ and $Y$, there is a natural isomorphism

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y).$$

**Solution.** Recall that the Cartesian product is the product in $\text{Top}_*$

$$(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0))$$

as well as in the homotopy category $\text{Ho}(\text{Top}_*)$. Therefore the natural map

$$\pi_n(X \times Y) = [S^n, X \times Y, \ast] \xrightarrow{\sim} [S^n, X, \ast] \times [S^n, Y, \ast] = \pi_n(X) \times \pi_n(Y)$$

is an isomorphism (in $\text{Set}_*$).

**Remark.** This argument works for arbitrary products, not just finite:

$$\varphi: \pi_n \left( \prod_{\alpha} X_{\alpha} \right) \xrightarrow{\sim} \prod_{\alpha} \pi_n(X_{\alpha}).$$

Moreover, the isomorphism $\varphi$ is an isomorphism of groups, since each of its coordinates

$$\pi_n \left( \prod_{\alpha} X_{\alpha} \right) \xrightarrow{\varphi} \prod_{\alpha} \pi_n(X_{\alpha}) \xrightarrow{\sim} \pi_n(X_{\beta})$$

is a group homomorphism, namely $\pi_n(p_{\beta})$ induced by the projection $p_{\beta}: \prod_{\alpha} X_{\alpha} \to X_{\beta}.$
Problem 3. (May § 9.6 Problem 1) Let $X$ and $Y$ be pointed spaces, and $n \geq 2$.

a. Show that the map $j_* : \pi_n(X \times Y) \to \pi_n(X \times Y, X \vee Y)$ is zero.

Solution. Consider the natural isomorphisms

$$\pi_n(X \times Y) \xrightarrow{(p_X^*, p_Y^*)} \pi_n(X) \times \pi_n(Y) \xleftarrow{\cong} \pi_n(X) \oplus \pi_n(Y)$$

where the last step comes from the fact that $\pi_n(X)$ and $\pi_n(Y)$ are abelian groups ($n \geq 2$). One readily checks that the inverse isomorphism is

$$\pi_n(X) \oplus \pi_n(Y) \xrightarrow{(\iota_X^*, \iota_Y^*)} \pi_n(X \times Y)$$

where $\iota_X : X \to X \times Y$ is the “slice inclusion” $\iota_X(x) = (x, y_0)$ and likewise for $\iota_Y$. Therefore, any element $\theta \in \pi_n(X \times Y)$ can be (uniquely) written as a sum $\theta = \theta_X + \theta_Y$ with $\theta_X \in \text{im } \iota_X^*$ and $\theta_Y \in \text{im } \iota_Y^*$.

Any element $\theta_X \in \text{im } \iota_X^*$ is represented by a map $D^n \to X \times Y$ whose image is contained in $X \times \{y_0\} \subseteq X \vee Y$, which implies $j_*(\theta_X) = 0 \in \pi_n(X \times Y, X \vee Y)$, and likewise $j_*(\theta_Y) = 0$. We conclude

$$j_*(\theta) = j_*(\theta_X + \theta_Y)$$

$$= j_*(\theta_X) + j_*(\theta_Y)$$

$$= 0 + 0$$

$$= 0$$

using the fact that $\pi_n(X \times Y, X \vee Y)$ is a group (as $n \geq 2$) and $j_*$ is a group homomorphism. \qed
b. Show that there is an isomorphism

$$\pi_n(X \vee Y) \cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$

**Solution.** Note that the slice inclusions $\iota_X$ and $\iota_Y$ factor through the subspace $X \vee Y \subseteq X \times Y$:

$$X \xleftarrow{\iota_X} X \vee Y \xrightarrow{i} X \times Y.$$ 

This provides a section of the map $i_*$ as follows:

$$\pi_n(X \vee Y) \xrightarrow{i_*} \pi_n(X \times Y) \xrightarrow{(\iota_X, \iota_Y)} \pi_n(X) \oplus \pi_n(Y).$$

Hence the long exact sequence

$$\pi_{n+1}(X \times Y) \xrightarrow{j_*} \pi_{n+1}(X \times Y, X \vee Y) \xrightarrow{\partial} \pi_n(X \vee Y) \xrightarrow{i_*} \pi_n(X \times Y) \xrightarrow{j_*} \pi_n(X \times Y, X \vee Y)$$

breaks down to a split short exact sequence

$$0 \longrightarrow \pi_{n+1}(X \times Y, X \vee Y) \xrightarrow{\partial} \pi_n(X \vee Y) \xrightarrow{i_*} \pi_n(X \times Y) \longrightarrow 0$$

which yields the isomorphism

$$\pi_n(X \vee Y) \cong \pi_n(X \times Y) \oplus \pi_{n+1}(X \times Y, X \vee Y)$$

$$\cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$
Problem 4. (May § 9.4 Lemma) Let $n \geq 2$ and consider the $n$-dimensional real projective space $\mathbb{R}P^n$. Show that the following holds: $\pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}/2$ and $\pi_k(\mathbb{R}P^n) \simeq \pi_k(S^n)$ for all $k \geq 2$.

Solution. Recall that $\mathbb{R}P^n$ is obtained as the quotient

$$\mathbb{R}P^n = S^n/O(1)$$

of the sphere by the free action of the group $O(1) = \{-1,1\} \subset \mathbb{R}^\times$. The quotient map $p: S^n \to \mathbb{R}P^n$ is thus a two-sheeted covering, and is in fact the universal cover of $\mathbb{R}P^n$, since $S^n$ is simply-connected for $n \geq 2$. Therefore $\pi_1(\mathbb{R}P^n)$ is isomorphic to $O(1) \simeq \mathbb{Z}/2$.

For $k \geq 2$, the covering map $p$ induces an isomorphism $p_*: \pi_k(S^n) \xrightarrow{\cong} \pi_k(\mathbb{R}P^n)$. \qed
Problem 5. (May § 9.6 Problem 2) Let $n \geq 3$.

a. Compute the group $\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1})$.

Solution. The standard inclusion $i: \mathbb{R}P^{n-1} \to \mathbb{R}P^n$ lifts to universal covers to the standard inclusion $\tilde{i}: S^{n-1} \to S^n$, as illustrated in the commutative diagram

$$
\begin{array}{ccc}
S^{n-1} & \longrightarrow & S^n \\
p & \downarrow & \downarrow p \\
\mathbb{R}P^{n-1} & \longrightarrow & \mathbb{R}P^n.
\end{array}
$$

Note that $\tilde{i}: S^{n-1} \to S^n$ is null-homotopic, as $\pi_{n-1}(S^n) = 0$. For any $k \geq 2$, applying $\pi_k$ to this diagram yields

$$
\begin{array}{ccc}
\pi_k(S^{n-1}) & \xrightarrow{\tilde{i}_* = 0} & \pi_k(S^n) \\
p_* & \simeq & \simeq & p_* \\
\pi_k(\mathbb{R}P^{n-1}) & \xrightarrow{i_*} & \pi_k(\mathbb{R}P^n)
\end{array}
$$

from which we conclude $i_* = 0$. Hence the long exact sequence

$$
\pi_n(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \pi_n(\mathbb{R}P^n) \xrightarrow{j_*} \pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \xrightarrow{\partial} \pi_{n-1}(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \pi_{n-1}(\mathbb{R}P^n)
$$

breaks down to a short exact sequence

$$
0 \longrightarrow \pi_n(\mathbb{R}P^n) \xrightarrow{j_*} \pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \xrightarrow{\partial} \pi_{n-1}(\mathbb{R}P^{n-1}) \longrightarrow 0
$$

which is automatically split, since $\pi_{n-1}(\mathbb{R}P^{n-1}) \simeq \mathbb{Z}$ is a projective $\mathbb{Z}$-module. This yields the isomorphism

$$
\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \simeq \pi_n(\mathbb{R}P^n) \oplus \pi_{n-1}(\mathbb{R}P^{n-1}) \\
\simeq \pi_n(S^n) \oplus \pi_{n-1}(S^{n-1}) \\
\simeq \mathbb{Z} \oplus \mathbb{Z}. \quad \square
$$
b. Deduce that the quotient map of pairs

\[ q: (\mathbb{R}P^n, \mathbb{R}P^{n-1}) \to (\mathbb{R}P^n/\mathbb{R}P^{n-1}, *) \]

does not induce an isomorphism on homotopy groups.

**Solution.** Recall that the standard CW-structure on \( \mathbb{R}P^n \) has one cell in each dimension \( 0, 1, \ldots, n \) and \( \mathbb{R}P^{n-1} \) is the \((n - 1)\)-skeleton of \( \mathbb{R}P^n \). Therefore we have a homeomorphism

\[ \mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n. \]

The quotient map \( q \) cannot induce an isomorphism on the \( n \)th relative homotopy groups, as the two groups are non-isomorphic:

\[ \pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z} \]
\[ \pi_n(\mathbb{R}P^n/\mathbb{R}P^{n-1}, *) \cong \pi_n(S^n, *) \cong \mathbb{Z} \not\cong \mathbb{Z} \oplus \mathbb{Z}. \]