Problem 3. (Whitehead products) For each $n \geq 1$, consider the sphere $S^n$ with its CW-structure having one 0-cell and one $n$-cell. For any positive integers $p, q \geq 1$, the product $S^p \times S^q$ inherits a CW-structure with four cells, in dimensions 0, $p$, $q$, and $p + q$ respectively. The $(p + q - 1)$-skeleton of $S^p \times S^q$ is $S^p \vee S^q$ so that the attaching map of the top cell has the form

$$w : S^{p+q-1} \to S^p \vee S^q.$$ 

For any pointed space $X$, precomposition by $w$ defines an operation

$$\pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X)$$

called the **Whitehead product**, denoted by brackets $[\alpha, \beta] \in \pi_{p+q-1}(X)$.

a. For $p = q = 1$, the Whitehead product takes the form $\pi_1(X) \times \pi_1(X) \to \pi_1(X)$. What is this map?

b. More generally, for $p = 1$ and $q \geq 1$, describe the Whitehead product $\pi_1(X) \times \pi_q(X) \to \pi_q(X)$.

c. Show that a path-connected H-space (c.f. Homework 3 Problem 1) has trivial Whitehead products.

Remark. There are annoying sign conventions involved in answering parts (a) and (b) carefully.