Problem 1 Evaluate each integral.

a) $\int xe^{2x}dx$ $\qquad u = x,\ du = dx$

$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$ $\qquad dv = e^{2x}dx,\ \ v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$

b) $\int \ln x dx$ $\qquad u = \ln x,\ \ du = \frac{1}{x}dx$

$= x\ln x - \int x \cdot \frac{1}{x}dx$ $\qquad dv = dx,\ \ v = x$

$= x\ln x - x + C.$

c) $\int \tan^3 x \sec^3 x dx = \int \tan^2 x \sec^2 x \tan x \sec x dx$

$= \int (\sec^2 x - 1)\sec^2 x \sec x \tan x dx$ $\qquad u = \sec x, \quad du = \sec x \tan x dx$

$= \int (u^2 - 1)u^2 du = \int u^4 - u^2 du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$

d) $\int \frac{x^3 - 2}{x^3 + x}dx = \int 1 - \frac{x+2}{x^3+x}dx$, using long division

$= \int 1 - \frac{x+2}{x(x^2+1)}dx$

Partial Fractions Decompositon:

$\frac{x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow x + 2 = A(x^2+1) + x(Bx+C).$

$x = 0 \Rightarrow 2 = A,$

$x + 2 = 2(x^2+1) + Bx^2 + Cx = (2+B)x^2 + Cx + 2 \Rightarrow 2 + B = 0,\ C = 1 \Rightarrow B = -2,\ C = 1.$

So, $\int 1 - \frac{x+2}{x(x^2+1)}dx = \int 1 - \frac{2}{x} - \frac{-2x+1}{x^2+1}dx$

$= \int 1 - \frac{2}{x} + \frac{2x}{x^2+!} - \frac{1}{x^2+1}dx$

$= x - 2\ln|x| + \ln|x^2+1| - \tan^{-1}x + C.$

e) $\int \frac{dx}{x^2 - 2x - 3}$

Partial Fractions Decompositon:

$\frac{1}{x^2 - 2x - 3} = \frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + B(x-3).$

$x = 3 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}.$

$x = -1 \Rightarrow 1 = -4B \Rightarrow B = -\frac{1}{4}$

So, $\int \frac{dx}{x^2 - 2x - 3} = \int \frac{1}{4}\frac{1}{x-3} - \frac{1}{4}\frac{1}{x+1}dx = \frac{1}{4}\ln|x-3| - \frac{1}{4}\ln|x+1| + C.$

f) $\int x\tan^{-1}x dx = \int \tan^{-1}x \cdot x dx$ $\qquad u = \tan^{-1}x,\ du = \frac{1}{1+x^2}dx$

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$ $\qquad dv = xdx,\ v = \frac{1}{2}x^2$

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}dx$

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C.$

g) $\int \frac{1}{x^2\sqrt{4x^2-9}}dx = \int \frac{1}{x^2\sqrt{(2x)^2 - 3^2}}dx$ $\qquad 2x = 3\sec\theta \Rightarrow \sec\theta = \frac{2x}{3},$

$x = \frac{3}{2}\sec\theta \Rightarrow dx = \frac{3}{2}\sec\theta\tan\theta d\theta$

$= \int \frac{1}{(9/4\sec^2\theta)(3\tan\theta)}\frac{3}{2}\sec\theta\tan\theta d\theta$

$= \int \frac{2}{9}\frac{1}{\sec\theta}d\theta = \frac{2}{9}\int\cos\theta d\theta$

$= \frac{2}{9}\sin\theta + C = \frac{2}{9}\frac{\sqrt{4x^2-9}}{2x} + C$

$= \frac{\sqrt{4x^2-9}}{9x} + C.$

Problem 2 Find the improper integrals, or determine that they diverge.

a) $\int_{-\infty}^{\infty} \frac{x}{x^2+4} dx = \lim_{a \to -\infty} \int_a^0 \frac{x}{x^2+4} dx + \lim_{b \to \infty} \int_0^b \frac{x}{x^2+4} dx \qquad u = x^2+1, \;\; du = 2xdx$

$= \lim_{a \to -\infty} \int_{a^2+1}^{1} \frac{1}{2} \frac{1}{u} du + \lim_{b \to \infty} \int_1^{b^2+1} \frac{1}{2} \frac{1}{u} du$

$= \lim_{a \to -\infty} \frac{1}{2} \ln|u| \big|_{a^2+1}^{1} + \lim_{b \to \infty} \frac{1}{2} \ln|u| \big|_1^{b^2+1}$

$= \lim_{a \to -\infty} \frac{1}{2} \ln 1 - \frac{1}{2} \ln(a^2+1) + \lim_{b \to \infty} \frac{1}{2} \ln(b^2+1) - \frac{1}{2} \ln 1$

diverges, since $\lim_{a \to -\infty} \frac{1}{2} \ln(a^2+1) = \infty$.

b) $\int_0^{\infty} xe^{-2x} dx = \lim_{b \to \infty} \int_0^b xe^{-2x} dx$

$= \lim_{b \to \infty} -\frac{1}{2} xe^{-2x} \big|_0^b + \frac{1}{2} \int_0^b e^{-2x} dx$

$= \lim_{b \to \infty} -\frac{1}{2} xe^{-2x} \big|_0^b - \frac{1}{4} e^{-2x} \big|_0^b = \lim_{b \to \infty} - \frac{1}{2} \frac{b}{e^{2b}} - \frac{1}{4} \frac{1}{e^{2b}} + \frac{1}{4}$

$= \lim_{b \to \infty} - \frac{1}{2} \frac{1}{2e^{2b}} - \frac{1}{4} \frac{1}{e^{2b}} + \frac{1}{4}$

$= \frac{1}{4}.$

c) $\int_2^4 \frac{1}{(x-2)^2} dx = \lim_{a \to 2^+} \int_a^4 \frac{1}{(x-2)^2} dx = \lim_{a \to 2^+} - \frac{1}{x-2} \Big|_a^4$

$= \lim_{a \to 2^+} - \frac{1}{4-2} + \frac{1}{a-2} = \infty$, diverges.

Problem 3 Use the Comparison test to determine whether the integral converges or diverges.

a. $\int_2^\infty \frac{dx}{\sqrt[4]{x^3-2}}$

$0 \le \frac{1}{\sqrt[4]{x^3}} \le \frac{1}{\sqrt[4]{x^3-2}}$ for $x \ge 2$.

Since $\int_2^\infty \frac{dx}{\sqrt[4]{x^3}}$ diverges (you need to check it), $\int_2^\infty \frac{dx}{\sqrt[4]{x^3-2}}$ diverges by the Comparison Test.

b. $\int_1^\infty \frac{dx}{x+e^x}$.

$0 \le \frac{1}{x+e^x} \le \frac{1}{e^x}$ for $x \ge 1$.

Since $\int_1^\infty e^{-x}dx$ converges (check it), $\int_1^\infty \frac{dx}{x+e^x}$ converges by the Comparison Test.

Problem 4 Solve the following differential equation:

$$y'(t) = -3y(t) \ : \ y(0) = 2$$

$y'(t) = -3y(t) \Rightarrow \frac{y'(t)}{y(t)} = -3$

$\Rightarrow \int \frac{y'(t)}{y(t)} dt = \int -3dt$

$\Rightarrow \ln|y(t)| = -3t + C$

$\Rightarrow |y(t)| = e^{-3t+C} = e^{3C}e^{-3t} = Ae^{-3t}$

$\Rightarrow y(t) = Be^{-3t}$.

$y(0) = 2 \Rightarrow B = 2$.

So, $y(t) = 2e^{-3t}$.