Problem 1 Determine whether each sequence $\{a_n\}$ converges of diverges. If it converges, find the limit.

(a) $a_n = \frac{2n+3}{n-1}$

$$\lim_{n\to\infty} \frac{2n+3}{n-1} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{1-\frac{1}{n}} = \frac{2}{1} = 2$$

(b) $a_n = (-1)^n$

Note $\{a_n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \cdots\}$ So, $\lim_{n\to\infty} a_n$ diverges.

OR, $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1)^{2n} = 1$, but $\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} (-1)^{2n+1} = -1 \Rightarrow$ diverges.

(c) $a_n = \frac{n}{\ln n}$

$$\lim_{x\to\infty} \frac{x}{\ln x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{(x)'}{(\ln x)'} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Hence, $\lim_{n\to\infty} \frac{n}{\ln n} = \infty$, diverges.

(d) $a_n = \frac{10+\sin n}{n}$

$$-1 \le \sin n \le 1 \Rightarrow 9 \le 10 + \sin n \le 11$$

$$\Rightarrow \frac{9}{n} \le \frac{10+\sin n}{n} \le \frac{11}{n}$$

Since $\lim_{n\to\infty} \frac{9}{n} = 0 = \lim_{n\to\infty} \frac{11}{n}$, $\lim_{n\to\infty} \frac{10+\sin n}{n} = 0$ by the Squeeze Theorem.

Problem 2 Determine whether the given series converges or diverges.

(a) $\sum_{k=1}^{\infty} \frac{1}{k^{1.5}}$

p-series with $p=1.5$ (>1) $\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^{1.5}}$ converges.

(b) $\sum_{k=1}^{\infty} \frac{1}{k^{0.5}}$

p-series with $p=0.5$ (<1) $\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^{0.5}}$ diverges.

Problem 3 Find the sum of the series.

(a) $\sum_{k=1}^{\infty} \frac{3^{k+1}}{5^k} = \frac{3^2}{5^1} + \frac{3^3}{5^2} + \frac{3^4}{5^3} + \cdots$

Geometric Series with the first term $\frac{3^2}{5}$ and ratio $\frac{3}{5}$.

Since the ratio is less than 1, the given geometric series converges

and $\sum_{k=1}^{\infty} \frac{3^{k+1}}{5^k} = \frac{3^2/5}{1-3/5} = \boxed{\frac{9}{2}}$.

(b) $\sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right)$: telescoping sum.

$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+2}\right)$

$= \lim_{n\to\infty} \frac{1}{2}\Big[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots$

$+ \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)\Big]$

$= \lim_{n\to\infty} \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2}\right) = \frac{3}{4}$

Problem 4 (a) State the k-th term divergence test.

Hypothesis:

$\lim_{k\to\infty} a_k \neq 0$

Conclusion:

$\sum_{k=1}^{\infty} a_k$ diverges.

(b) Determine whether $\sum_{k=1}^{\infty} 2^{-\frac{\ln k}{k}}$ converges or diverges.

Note that $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

$\lim_{k\to\infty} 2^{-\frac{\ln k}{k}} = 2^{-0} = 2^0 = 1 \neq 0$.

Hence the series diverges by the k-th-Term Divergence Test.

Problem 5 (a) State the integral test.

Hypothesis: Let $f(x)$ be

① continuous

② decreasing

③ $f(x) \geq 0$ for all $x \geq 1$

and ④ $f(k) = a_k$, $k = 1, 2, \cdots$.

Conclusion:

$\int_1^\infty f(x)dx$ and $\sum_{k=1}^{\infty} a_k$ either

both converges

or both diverges.

(b) Determine the series $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ is convergent or divergent.

$$f(x) = \frac{1}{x \ln x}$$

$\Rightarrow$ For all $x \geq 2$, $f(x)$ is continuous,

decreasing ($f'(x) = -\frac{1+\ln x}{(x \ln x)^2} < 0$, $x \geq 2$)

and $f(x) \geq 0$

$$\int_2^\infty \frac{1}{x \ln x} dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln x} dx \qquad \begin{matrix} u = \ln x \\ du = \frac{1}{x} dx \end{matrix}$$

$$= \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u} du = \lim_{b \to \infty} \ln|u| \Big|_{\ln 2}^{\ln b} = \lim_{b \to \infty} \ln|\ln b| - \ln|\ln 2| = \infty.$$

Hence, $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges by the Integral Test.

Problem 6 (a) State the Comparison test.

Hypothesis: Suppose that

$$0 \le a_k < b_k \text{ for all } k$$

Conclusion:

If $\sum_{k=1}^{\infty} b_k$ converges, $\sum_{k=1}^{\infty} a_k$ converges.

If $\sum_{k=1}^{\infty} a_k$ diverges, $\sum_{k=1}^{\infty} b_k$ diverges.

(b) Determine whether the given series converges or diverges.

i) $\sum_{k=2}^{\infty} \frac{k^2}{k^3 - 3} \sim \sum_{k=2}^{\infty} \frac{k^2}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k}$

$$0 \le \frac{k^2}{k^3} < \frac{k^2}{k^3 - 3} \quad \text{for all } k \ge 2$$

$\sum_{k=2}^{\infty} \frac{1}{k}$ diverges (Harmonic series)

Hence $\sum_{k=2}^{\infty} \frac{k^2}{k^3-3}$ diverges by the Comparison Test.

ii) $\sum_{k=1}^{\infty} \frac{1}{k + 2^k} \sim \sum_{k=1}^{\infty} \frac{1}{2^k}$

$$0 \le \frac{1}{k+2^k} \le \frac{1}{2^k} \quad \text{for all } k \ge 1.$$

Since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges (geometric series with $r = \frac{1}{2}$),

$\sum_{k=1}^{\infty} \frac{1}{k+2^k}$ converges by the Comparison Test.

Problem 7 (a) State the Limit Comparison test.

Hypothesis: Suppose that

$a_k, b_k > 0$

$\lim_{k\to\infty} \frac{a_k}{b_k} = L \quad (L \neq 0, L \neq \infty)$

Conclusion:

Either $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge

or they both diverge.

(b) Determine whether the given series converges or diverges.

i) $\sum_{k=1}^{\infty} \frac{\sqrt{k}+1}{4k+3}$ $\sim \sum_{k=1}^{\infty} \frac{\sqrt{k}}{4k} = \sum_{k=1}^{\infty} \frac{1}{4\sqrt{k}}$

Let $a_k = \frac{\sqrt{k}+1}{4k+3}$ and $b_k = \frac{1}{4\sqrt{k}}$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{(\sqrt{k}+1)(4\sqrt{k})}{4k+3} = \lim_{k\to\infty} \frac{4k+4\sqrt{k}}{4k+3}$

$= \lim_{k\to\infty} \frac{4+\frac{4\sqrt{k}}{k}}{4+\frac{3}{k}} = \lim_{k\to\infty} \frac{4+\frac{4}{\sqrt{k}}}{4+\frac{3}{k}} = 1.$

Since $\sum_{k=1}^{\infty} \frac{1}{4\sqrt{k}}$ diverges ($p=\frac{1}{2}$), $\sum_{k=1}^{\infty} \frac{\sqrt{k}+1}{4k+3}$ diverges.

ii) $\sum_{k=1}^{\infty} \frac{k^2+4}{k^5+3k+1}$ $\sim \sum_{k=1}^{\infty} \frac{k^2}{k^5} = \sum_{k=1}^{\infty} \frac{1}{k^3}$

$a_k = \frac{k^2+4}{k^5+3k+1}$, $b_k = \frac{1}{k^3}$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{(k^2+4)k^3}{k^5+3k+1} = \lim_{k\to\infty} \frac{k^5+4k^3}{k^5+3k+1} = 1$

Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges ($p=3>1$),

$\sum_{k=1}^{\infty} \frac{k^2+4}{k^5+3k+1}$ converges by the Limit Comparison Test.

Problem 8 (a) State the Alternating series test for an alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ with $a_k > 0$.

Hypothesis: ① $\lim_{k\to\infty} a_k = 0$.

② $0 < a_{k+1} \le a_k$ for all $k \geq 1$.

Conclusion: $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$, converges.

(b) Determine whether the series converges or diverges.

i) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ $\quad a_k = \frac{1}{k}$

$\lim_{k\to\infty} a_k = 0$

$0 < \frac{a_{k+1}}{a_k} = \frac{k}{k+1} < 1$ for all $k \geq 1$

$\Rightarrow$ the series converges

ii) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{3^k}$, $\quad a_k = \frac{k}{3^k}$

$\lim_{k\to\infty} \frac{k}{3^k} \ \left(\frac{\infty}{\infty}\right)$

$= \lim_{k\to\infty} \frac{1}{\ln 3 \cdot 3^k} = 0$,

$0 < \frac{a_{k+1}}{a_k} = \frac{k+1}{3^{k+1}} \cdot \frac{3^k}{k} = \frac{k+1}{3k} < \frac{k+2k}{3k} = 1,\ k \geq 1$

$\Rightarrow 0 < a_{k+1} < a_k$

Hence, the series converges

iii) $\sum_{k=6}^{\infty}(-1)^{k+1}e^{1/k}$

$\lim_{k\to\infty} e^{1/k} = e^0 = 1 \neq 0$ $\left(\Rightarrow \lim_{k\to\infty} (-1)^{k+1} e^{1/k} \neq 0\right)$

Hence, the series diverges by k-th term Divergence Test.

Problem 9 (a) State the Ratio test: Consider $\sum_{k=1}^{\infty} a_k$ with $a_k \neq 0$ for all k.

Let $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L$. Then,

① if $L<1$, the series converges absolutely (so, does the series)

② if $L>1$ (or $L=\infty$), the series diverges.

③ if $L=1$, the test is useless (inconclusive.)

(b) Determine whether the series is convergent or divergent.

i) $\sum_{k=1}^{\infty} \frac{k^2}{10^k}$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{(k+1)^2}{10^{k+1}} \cdot \frac{10^k}{k^2} = \lim_{k\to\infty} \frac{(k+1)^2}{10k^2}$$

$$= \frac{1}{10}$$

$L = \frac{1}{10}$ (<1) So, the series converges.

ii) $\sum_{k=1}^{\infty} \frac{(2k+1)!}{3^k}$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{(2k+3)!}{3^{k+1}} \cdot \frac{3^k}{(2k+1)!}$$

$$= \lim_{k\to\infty} \frac{(2k+1)!\,(2k+2)(2k+3)}{3\,(2k+1)!}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+3)}{3}$$

$$= \infty.$$

So the series diverges.

Note that

$$a_{k+1} = \frac{(2(k+1)+1)!}{3^{k+1}} = \frac{(2k+3)!}{3^{k+1}}$$

iii) $\sum_{k=1}^{\infty} (-1)^k \frac{4^k}{k!}$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| (-1)^{k+1} \cdot \frac{4^{k+1}}{(k+1)!} \cdot (-1)^k \cdot \frac{k!}{4^k} \right|$$

$$= \lim_{k\to\infty} \frac{4}{(k+1)} = 0.$$

So, the series converges absolutely (i.e., $\sum_{k=1}^{\infty} \left| (-1)^k \cdot \frac{4^k}{k!} \right| = \sum_{k=1}^{\infty} \frac{4^k}{k!}$ converges)

Thus, $\sum_{k=1}^{\infty} (-1)^k \frac{4^k}{k!}$ converges.

iv) $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k}$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| (-1)^{k+1} \cdot \frac{2^{k+1}}{k+1} \cdot (-1)^k \cdot \frac{k}{2^k} \right|$$

$$= \lim_{k\to\infty} \frac{2^{k+1} \cdot k}{2^k (k+1)} = \lim_{k\to\infty} \frac{2k}{k+1} = 2$$

Since $L = 2 > 1$, the series diverges.

Note that $\lim_{k\to\infty} \frac{2^k}{k} \quad \left(\frac{\infty}{\infty}\right)$

$$= \lim_{k\to\infty} \frac{\ln 2 \cdot 2^k}{1}$$

$= \infty. \; (\neq 0) \Rightarrow$ the series diverges by k-th term divergence test

Problem 10 (a) State the Root test: Consider $\sum_{k=1}^{\infty} a_k$.

Let $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L$.

Then,

① if $L<1$, the series converges absolutely.

② if $L>1$ (or $L=\infty$), the series diverges.

③ if $L=1$, No conclusion.

(b) Determine whether the series is convergent or divergent.

i) $\sum_{k=1}^{\infty}\left(\frac{2k}{3k+1}\right)^k$

$$0 \le a_k = \left(\frac{2k}{3k+1}\right)^k \Rightarrow |a_k| = \left(\frac{2k}{3k+1}\right)^k$$

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{2k}{3k+1}\right)^k} = \lim_{k\to\infty} \frac{2k}{3k+1} = \frac{2}{3} \ (<1)$$

Hence, the series converges.

ii) $\sum_{k=1}^{\infty}\left(\frac{2+\ln k}{k}\right)^k$

$$0 \le a_k = \left(\frac{2+\ln k}{k}\right)^k \Rightarrow |a_k| = \left(\frac{2+\ln k}{k}\right)^k$$

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{2+\ln k}{k} \quad \frac{\infty}{\infty}$$

$$= \lim_{k\to\infty} \frac{1/k}{1} = 0.$$

Hence, the series converges.

Problem 11 Determine whether the series converges absolutely, converges conditionally or diverges.

(a) $\sum_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

Since the given series is an alternating series with $a_k = \frac{1}{\sqrt{k}}$ ($\lim_{k\to\infty} a_k = 0$, $0 < a_{k+1} < a_k$), it converges

But $\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{\sqrt{k}}\right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges, since it is a p-series with $p = \frac{1}{2}$. Hence, the series converges conditionally.

(b) $\sum_{k=1}^{\infty} \frac{\sin k}{k^{3/2}}$

$0 \le \left|\frac{\sin k}{k^{3/2}}\right| \le \frac{1}{k^{3/2}}$ for all $k \ge 1$

Since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges (p-series with $p = \frac{3}{2}$),

$\sum_{k=1}^{\infty} \left|\frac{\sin k}{k^{3/2}}\right|$ converges, i.e, the series converges absolutely.

(c) $\sum_{k=1}^{\infty} (-1)^k \frac{k+1}{k^2}$: alternating series with $a_k = \frac{k+1}{k^2}$

$\lim_{k\to\infty} \frac{k+1}{k^2} = 0$, and $0 < \frac{a_{k+1}}{a_k} = \frac{(k+2)\cdot k^2}{(k+1)^2(k+1)} = \frac{k^3+2k^2}{k^3+3k^2+3k+1} < 1$

Hence, the series converges

But, $\sum_{k=1}^{\infty} \left|(-1)^k \cdot \frac{k+1}{k^2}\right| = \sum_{k=1}^{\infty} \frac{k+1}{k^2}$ diverges by the Comparison Test

; $0 \le \frac{k}{k^2} \le \frac{k+1}{k^2}$ for all $k \ge 1$

$\sum_{k=1}^{\infty} \frac{k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Hence, the series $\sum_{k=1}^{\infty} (-1)^k \frac{k+1}{k^2}$ converges conditionally.

Problem 12 Suppose that $\sum_{k=1}^{\infty} a_k$ is an infinite series with the property that $\sum_{k=1}^{n} a_k = \frac{\ln n}{n}$ for all positive integers n.
Does $\sum_{k=1}^{\infty} a_k$ converge? Justify your answer.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

$$= \lim_{n\to\infty} \frac{\ln n}{n}$$

$$= \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

Hence, $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} a_k = 0$.

Problem 13 Show that if $a_k > 0$ for all k and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} a_k^2$ converges.

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Hence we can find sufficiently large N so that $a_k < 1$ for all $k \geq N$.

$\Rightarrow 0 < a_k^2 < a_k$ for all $k \geq N$.

By the Comparison Test, $\sum_{k=N}^{\infty} a_k^2$ converges.

Hence, $\sum_{k=1}^{\infty} a_k^2$ converges.