Problem 1 Find the radius of convergence and the interval of convergence of each power series.

(a) $\sum_{k=0}^{\infty}(x-3)^k$

$$\lim_{k\to\infty}\left|\frac{(x-3)^{k+1}}{(x-3)^k}\right| = \lim_{k\to\infty}|x-3| = |x-3| < 1$$

So, the radius of convergence is 1.

$|x-3| < 1 \iff -1 < x-3 < 1 \iff 2 < x < 4$

Endpoints: $x = 2$; $\Sigma\,(-1)^k$ diverges

$x = 4$; $\Sigma\,(1)^k$ diverges

Hence, the interval of convergence is $(-2, 4)$.

(b) $\sum_{k=0}^{\infty}(3x+1)^k$

$$\lim_{k\to\infty}\left|\frac{(3x+1)^{k+1}}{(3x+1)^k}\right| = |3x+1| < 1$$

So, the radius of convergence is 1.

$|3x+1| < 1 \iff -1 < 3x+1 < 1 \iff -\frac{2}{3} < x < 0$.

Endpoints: $x = -\frac{2}{3}$; $\Sigma\,(-1)^k$ diverges

$x = 0$; $\Sigma\,(1)^k$ diverges

Hence, the interval of convergence is $(-\frac{2}{3}, 0)$

(c) $\sum_{k=0}^{\infty}\frac{k!}{(2k)!}x^k$

$$\lim_{k\to\infty}\left|\frac{(k+1)!}{(2(k+1))!}\cdot\frac{(2k)!}{k!}\cdot\frac{x^{k+1}}{x^k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!}{k!}\,\frac{(2k)!}{(2k+2)!}\,x\right|$$

$$= \lim_{k\to\infty}\frac{k+1}{(2k+1)(2k+2)}|x|$$

$$= 0\cdot|x| < 1 \quad \text{for all } x\in\mathbb{R}.$$

∴ Hence, the radius of convergence is $\infty$,

and $(-\infty, \infty)$ is the interval of convergence.

(d) $\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1}(2x)^k$

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1}}{4(k+1)^2-1} \cdot \frac{4k^2-1}{(-1)^k} \cdot \frac{(2x)^{k+1}}{(2x)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{4k^2-1}{4(k+1)^2-1} \right| \cdot |2x| = 1 \cdot |2x| < 1$$

So, the radius of convergence is 1.

$|2x| < 1 \iff -1 < 2x < 1 \iff -\frac{1}{2} < x < \frac{1}{2}$.

Endpoints ; $x = -\frac{1}{2}$ : $\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1}(-1)^k = \sum_{k=1}^{\infty} \frac{1}{4k^2-1}$ converges

(Limit Comparison Test)

$x = \frac{1}{2}$ : $\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1}(1)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1}$ converges

(Alternating Series Test).

Hence, $[-\frac{1}{2}, \frac{1}{2}]$ is the interval of convergence.

(e) $\sum_{k=1}^{\infty} \frac{(x+4)^k}{k2^k}$

$$\lim_{k\to\infty} \left| \frac{1}{(k+1)2^{k+1}} \cdot \frac{k2^k}{1} \cdot \frac{(x+4)^{k+1}}{(x+4)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{k}{2(k+1)} (x+4) \right| = \lim_{k\to\infty} \frac{k}{2k+2} \cdot |x+4|$$

$$= \frac{1}{2}|x+4|$$

$\frac{1}{2}|x+4| < 1 \iff |x+4| < 2$

Hence, the radius of convergence is 2.

$|x+4| < 2 \iff -2 < x+4 < 2 \iff -6 < x < -2$

Endpoints ; $x = -6$ : $\sum_{k=1}^{\infty} \frac{(-2)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges

$x = -2$ ; $\sum_{k=1}^{\infty} \frac{(2)^k}{k \cdot 2^k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Hence, $[-6, -2)$ is the interval of convergence.

Problem 2 Find the power series of the form $\sum_{k=0}^{\infty} a_k x^k$ converging to each function. In each case, specify the radius of convergence of the power series.

(a) $\frac{2}{2-x}$

$$\frac{2}{2-x} = \frac{2}{2(1-x/2)} = \frac{1}{1-x/2}$$

( Geometric Sum with $a=1$, $r = x/2$ )

$$= \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k, \qquad \left|\frac{x}{2}\right| < 1$$

$\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$.

Hence, the radius of convergence is 2.

(b) $\frac{3}{(x-1)^2}$

Note that ① $\frac{3}{(x-1)^2} = \left(-\frac{1}{x-1}\right)' = \left(\frac{1}{1-x}\right)'$

② $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, $\quad |x| < 1$ ( $\Rightarrow$ radius of conv $= 1$ )

Hence, $$\frac{3}{(x-1)^2} = \left(\sum_{k=0}^{\infty} x^k\right)'$$

$$= \sum_{k=0}^{\infty} (x^k)' = \left(1 + \sum_{k=1}^{\infty} x^k\right)'$$

$$= (1)' + \sum_{k=1}^{\infty} (x^k)' = \sum_{k=1}^{\infty} k \cdot x^{k-1}$$

Since the radius of convergence remains the same under differentiation, the radius of convergence is 1.

(c) $\ln(1+x)$

Note that ① $\ln(1+x) + c = \int \frac{1}{1+x}\,dx$

② $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k, \quad |x|<1.$

Hence, $\ln(1+x) + c = \int \sum_{k=0}^{\infty} (-1)^k x^k \, dx$

$$= \sum_{k=0}^{\infty} \int (-1)^k x^k \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

Let $x=0$. Then $\ln(1) + c = 0 \Rightarrow c = 0$.

Thus, $\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$, and the radius of convergence is 1 since it remains the same under integration.

(d) $\tan^{-1}x$

Note that ① $\tan^{-1}x + c = \int \frac{1}{1+x^2}\,dx$

② $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k},$

for $|-x^2|<1$ ($\Rightarrow |x|<1$ )

Hence, $\tan^{-1}x + c = \int \sum_{k=0}^{\infty} (-1)^k x^{2k} \, dx$

$$= \sum_{k=0}^{\infty} (-1)^k \int x^{2k} \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

Let $x=0 \Rightarrow \tan^{-1}0 + c = 0 \Rightarrow c = 0$

$\therefore \tan^{-1}x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1},$

and the radius of convergence 1.

Problem 3 Find the Taylor series for $e^{3x}$ in $x + 1$.

$$\begin{array}{l|l}
f(x) = e^{3x} & f(-1) = e^{-3} \\
f'(x) = 3e^{3x} & f'(-1) = 3e^{-3} \\
f''(x) = 3^2 e^{3x} & f''(-1) = 3^2 e^{-3} \\
\vdots & \vdots \\
f^{(k)}(x) = 3^k e^{3x} & f^{(k)}(-1) = 3^k e^{-3}
\end{array}$$

Note that coefficient of $(x+1)^k$ in the Taylor series

is $\dfrac{f^{(k)}(-1)}{k!} = \dfrac{3^k e^{-3}}{k!}$

Hence, $e^{3x} = \boxed{\displaystyle\sum_{k=0}^{\infty} \frac{3^k e^{-3}}{k!} (x+1)^k}$

Problem 4 Find the Taylor series for $\sin x$ in $x - \pi$.

$$\begin{array}{l|l}
f(x) = \sin x & f(\pi) = \sin \pi = 0 \\
f'(x) = \cos x & f'(\pi) = \cos \pi = -1 \\
f''(x) = -\sin x & f''(\pi) = -\sin \pi = 0 \\
f'''(x) = -\cos x & f'''(\pi) = -\cos \pi = 1 \\
f^{(4)}(x) = \sin x & \vdots \\
\vdots &
\end{array}$$

$$\sin x = -\frac{1}{1!}(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \cdots$$

$$\left( = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} (x-\pi)^{2k+1} \right)$$

Problem 5 Evaluate each limit.

(a) $\lim_{x\to 0} \dfrac{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}-\cos x}{x^7 \sin x}$

$$= \lim_{x\to 0} \frac{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\cdots\right)}{x^7\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{x^8}{8!}+\frac{x^{10}}{10!}-\frac{x^{12}}{12!}+\cdots}{x^8\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots\right)}$$

$$= \lim_{x\to 0} \frac{x^8\left(-\frac{1}{8!}+\frac{x^2}{10!}-\frac{x^4}{12!}+\cdots\right)}{x^8\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{8!}+\frac{x^2}{10!}-\frac{x^4}{12!}+\cdots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots} = \boxed{-\frac{1}{8!}}$$

(b) $\lim_{x\to 0} \dfrac{\tan^{-1}x - x + \frac{x^3}{3}}{x^5 \cos x}$

$$= \lim_{x\to 0} \frac{\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots\right)-x+\frac{x^3}{3}}{x^5\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)}$$

$$= \lim_{x\to 0} \frac{\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots}{x^5\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{5}-\frac{x^2}{7}+\frac{x^4}{9}-\cdots}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots}$$

$$= \boxed{\frac{1}{5}}$$

Problem 6 Use the first three nonzero terms of a Taylor series to approximate $\cos\left(\frac{1}{100}\right)$. Estimate the error in the approximation.

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\cos\left(\frac{1}{100}\right) = \underbrace{1 - \frac{1}{2!}\left(\frac{1}{100}\right)^2 + \left(\frac{1}{4!}\right)\left(\frac{1}{100}\right)^4}_{\text{first 3 nonzero terms.}} - \frac{1}{6!}\left(\frac{1}{100}\right)^6 + \cdots \quad \text{; alternating series}$$

Hence, $\underline{\cos\left(\frac{1}{100}\right) \approx 1 - \frac{1}{2!}\left(\frac{1}{100}\right)^2 + \frac{1}{4!}\left(\frac{1}{100}\right)^4}$

Note that the series given here is an alternating series

Hence then $\underline{|\text{Error}| < \frac{1}{6!}\left(\frac{1}{100}\right)^6}$

Problem 7 Use the Taylor polynomial with $n = 4$ to approximate $e^{.1}$. Estimate the error in the approximation.

By Taylor's theorem,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + R_4(x)$$

with $R_4(x) = \frac{f^{(5)}(z)}{5!}x^5 = \frac{e^z}{5!}x^5$

for some $z$ between $x$ and $0$.

$$\Rightarrow e^{.1} \approx 1 + (0.1) + \frac{1}{2!}(0.1)^2 + \frac{1}{3!}(0.1)^3 + \frac{1}{4!}(0.1)^4,$$

$$|\text{Error}| = |R_4(0.1)| = \frac{e^z}{5!}(0.1)^5, \quad z \in (0, 0.1)$$

$$< \frac{e^{0.1}}{5!}(0.1)^5 < \boxed{\frac{3^{0.1}}{5!}(0.1)^5}$$

Note: ① the series here is not an alternating series.

② We assume that $2 < e < 3$.

Problem 8 Approximate $\int_0^{\frac{1}{10}} \frac{\sin x}{x}\,dx$ using the first three nonzero terms of a Taylor series, and estimate the error in the approximation.

$$\int_0^{1/10} \frac{\sin x}{x}\,dx = \int_0^{1/10} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x}\,dx$$

$$= \int_0^{1/10} \underbrace{1 - \frac{x^2}{3!} + \frac{x^4}{5!}}_{\text{first 3 terms}} - \frac{x^6}{7!} + \cdots\, dx$$

$$= x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots \Big|_0^{1/10}$$

$$= \frac{1}{10} - \frac{1}{10^3\cdot 3\cdot 3!} + \frac{1}{10^5\cdot 5\cdot 5!} - \frac{1}{10^7\cdot 7\cdot 7!} + \cdots$$

Hence, $\boxed{\int_0^{1/10} \frac{\sin x}{x} \approx \frac{1}{10} - \frac{1}{10^3\cdot 3\cdot 3!} + \frac{1}{10^5\cdot 5\cdot 5!}, \quad |\text{Error}| \le \frac{1}{10^7\cdot 7\cdot 7!}}$

Problem 9 Find the length of the parametric curve

$$x = 2t,\ \ y = t^3 + \frac{1}{3t},\ \ 1 \le t \le 2.$$

$$\text{Arc Length} = \int_1^2 \sqrt{(x')^2 + (y')^2}\,dt$$

$$= \int_1^2 \sqrt{(2)^2 + (3t^2 - \tfrac{1}{3}t^{-2})^2}\,dt$$

$$= \int_1^2 \sqrt{4 + 9t^4 - 2 + \tfrac{1}{9}t^{-4}}\,dt$$

$$= \int_1^2 \sqrt{9t^4 + 2 + \tfrac{1}{9}t^{-4}}\,dt$$

$$= \int_1^2 \sqrt{(3t^2 + \tfrac{1}{3}t^{-2})^2}\,dt$$

$$= \int_1^2 3t^2 + \tfrac{1}{3}t^{-2}\,dt = t^3 - \tfrac{1}{3}t^{-1}\Big|_1^2$$

$$= (8 - \tfrac{1}{6}) - (1 - \tfrac{1}{3}) = \boxed{\frac{43}{6}}$$

Problem 10 Find the length of the curve $y = \frac{3}{8}(x^{4/3} - 2x^{2/3}),\ 1 \le x \le 8.$

Parametric equation : $x = t$

$y = 3/8\,(t^{4/3} - 2t^{2/3}) \quad 1 \le t \le 8.$

$$\text{Arc Length} = \int_1^8 \sqrt{(x')^2 + (y')^2}\,dt$$

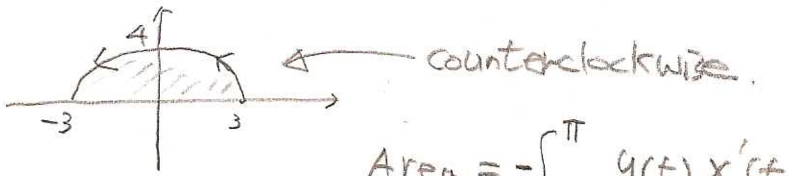
$$= \int_1^8 \sqrt{(1)^2 + \left(\tfrac{1}{2}(t^{1/3} - t^{-1/3})\right)^2}\,dt$$

$$= \int_1^8 \sqrt{\tfrac{1}{4}(t^{2/3} + 2 + t^{-2/3})}\,dt = \int_1^8 \sqrt{\tfrac{1}{4}(t^{1/3} + t^{-1/3})^2}\,dt$$

$$= \int_1^8 \tfrac{1}{2}(t^{1/3} + t^{-1/3})\,dt = \tfrac{3}{8}t^{4/3} + \tfrac{3}{4}t^{2/3}\Big|_1^8 = 9 - \tfrac{9}{8} = \boxed{\tfrac{63}{8}}$$

Problem 11 Find the area of the region between the parametric curve $y = 3\sin t$, $x = 4\cos t,\ 0 \le t \le \pi$, and the $x$-axis.

Note : $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$



$$\text{Area} = -\int_0^\pi y(t)\,x'(t)\,dt$$

$$= -\int_0^\pi 3\sin t\,(-4\sin t)\,dt$$

$$= 12\int_0^\pi \frac{1-\cos 2t}{2}\,dt, \qquad \sin^2 t = \frac{1-\cos 2t}{2}$$

$$= 6t - 3\sin 2t\Big|_0^\pi$$

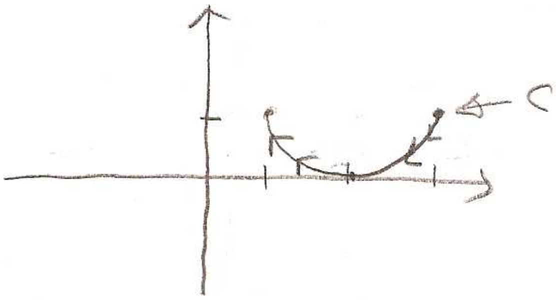
$$= \boxed{6\pi}$$

Problem 12 Let $C$ be the parametric curve

$$x(t) = 2 + \cos t,\ \ y(t) = 1 - \sin t,\ \ 0 \le t \le \pi.$$

(a) Sketch the curve.

| $t$ | $x = 2+\cos t$ | $y = 1-\sin t$ |
|---|---|---|
| $0$ | $3$ | $1$ |
| $\frac{\pi}{2}$ | $2$ | $0$ |
| $\pi$ | $1$ | $1$ |



(b) Find the direct relation between $x$ and $y$ by eliminating the parameter $t$.

$$\left.\begin{array}{l} x-2 = \cos t \\ y-1 = -\sin t \end{array}\right\} \Rightarrow \boxed{(x-2)^2 + (y-1)^2 = 1}$$

$$0 \le t \le \pi \Rightarrow -1 \le \cos t \le 1 \Rightarrow 1 \le x \le 3$$

$$0 \le \sin t \le 1 \Rightarrow 0 \le y \le 1$$

(c) Find the slope of the tangent line at $t = \frac{\pi}{4}$.

$$\text{slope} = \left.\frac{dy}{dx}\right|_{t=\pi/4} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi/4}$$

$$= \left.\frac{-\cos t}{-\sin t}\right|_{t=\pi/4} = 1.$$

(d) Find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. Is the curve concave up or concave at $t = \frac{\pi}{4}$? ← Ignore this problem!

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\cos t}{\sin t}\right) = \frac{d}{dx}(\cot t)$$

$$= \frac{(\cot t)'}{(2+\cos t)'} = \left.\frac{-\csc^2 t}{-\sin t}\right|_{t=\pi/4}$$

$$= \frac{(2/\sqrt{2})^2}{\sqrt{2}/2} = \frac{4/2}{\sqrt{2}/2} = 2\cdot\frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} > 0$$

→ concave upward

Problem 13 Let $C$ be the parametric curve

$$x(t) = t$$
$$y(t) = \frac{1}{3}t^{3/2} - t^{1/2}, \quad 1 \le t \le 4.$$

(a) Find the length of the curve.

$$\text{Arc Length} = \int_1^4 \sqrt{(x')^2 + (y')^2}\, dt$$

$$= \int_1^4 \sqrt{1 + \left(\tfrac{1}{2}t^{1/2} - \tfrac{1}{2}t^{-1/2}\right)^2}\, dt$$

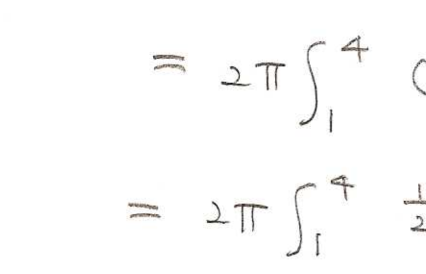
$$= \int_1^4 \sqrt{\tfrac{1}{4}t^{1/4} + \tfrac{1}{2} + \tfrac{1}{4}t^{-1/4}}\, dt = \int_1^4 \sqrt{\left(\tfrac{1}{2}t^{1/2} + \tfrac{1}{2}t^{-1/2}\right)^2}\, dt$$

$$= \int_1^4 \tfrac{1}{2}t^{1/2} + \tfrac{1}{2}t^{-1/2}\, dt = \tfrac{1}{3}t^{3/2} + t^{1/2}\Big|_1^4 = \left(\tfrac{8}{3} + 2\right) - \left(\tfrac{1}{3} + 1\right)$$

$$= \boxed{\tfrac{10}{3}}$$

(b) Find the area of the surface obtained by revolving the curve $C$ about the line $x = 1$.

$$\text{Surface Area} = 2\pi \int_1^4 (\text{Radius})(ds)$$



$$= 2\pi \int_1^4 |x-1| \cdot \sqrt{(x')^2 + (y')^2}\, dt$$

$$= 2\pi \int_1^4 |t-1| \cdot \left(\tfrac{1}{2}t^{1/2} + \tfrac{1}{2}t^{-1/2}\right) dt$$

$$= 2\pi \int_1^4 (t-1)\left(\tfrac{1}{2}t^{1/2} + \tfrac{1}{2}t^{-1/2}\right) dt \quad \left(\because t-1 \ge 0 \text{ for } 1 \le t \le 4\right)$$

$$= 2\pi \int_1^4 \tfrac{1}{2}t^{3/2} + \cancel{\tfrac{1}{2}t^{1/2}} - \cancel{\tfrac{1}{2}t^{1/2}} - \tfrac{1}{2}t^{-1/2}\, dt$$

$$= 2\pi \left[\tfrac{1}{5}t^{5/2} - t^{1/2}\right]_1^4$$

$$= 2\pi \left[\left(\tfrac{32}{5} - 2\right) - \left(\tfrac{1}{5} - 1\right)\right] = \boxed{\tfrac{52\pi}{5}}$$