Problem 1 Rewrite each polar curve as a rectangular equation.

(a) $r = 2$ $\quad r^2 = 2^2 \Rightarrow \boxed{x^2+y^2=4}$

(b) $r = 2\cos\theta$

$$r \cdot r = r(2\cos\theta) \Rightarrow r^2 = 2r\cos\theta$$
$$\Rightarrow x^2+y^2 = 2x \Rightarrow \boxed{(x-1)^2+y^2=1}$$

(c) $(r\sin^2\theta = \cos\theta) \times r$

$$\Rightarrow r^2\sin^2\theta = r\cos\theta \Rightarrow \boxed{y^2 = x}$$

(d) $r = 3\csc\theta$

$$\Rightarrow r\sin\theta = 3 \Rightarrow \boxed{y=3}$$

(e) $r = \dfrac{1}{\sin\theta+\cos\theta}$

$$\Rightarrow r\sin\theta + r\cos\theta = 1 \Rightarrow \boxed{y+x=1}$$

Problem 2 Let $r = 5\cos\theta,\ 0 \le \theta \le \frac{3\pi}{4}$.

(a) Find a parametric equation for the curve.

$$\left.\begin{aligned} x &= r\cos\theta = 5\cos\theta\cdot\cos\theta = 5\cos^2\theta \\ y &= r\sin\theta = 5\cos\theta\cdot\sin\theta \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x &= 5\cos^2\theta \\ y &= 5\cos\theta\cdot\sin\theta \end{aligned}}$$

(b) Find the slope of the tangent line at $\theta = \frac{\pi}{3}$.

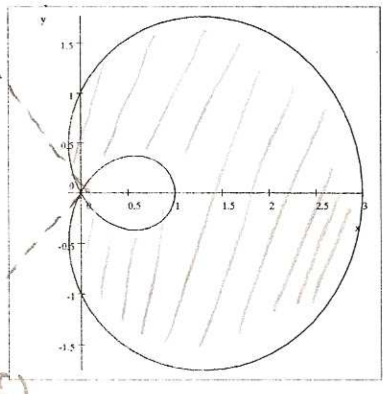
$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/3} = \left.\frac{-5\sin^2\theta+5\cos^2\theta}{-10\cos\theta\cdot\sin\theta}\right|_{\theta=\pi/3} = \boxed{\frac{\sqrt{3}}{3}}$$

(c) Find the length of the curve.

$$L = \int_0^{3\pi/4}\sqrt{r^2+(r')^2}\,d\theta = \int_0^{3\pi/4}\sqrt{25\cos^2\theta+25\sin^2\theta}\,d\theta$$
$$= \frac{3\pi}{4}\cdot 5 = \boxed{\frac{15\pi}{4}}$$

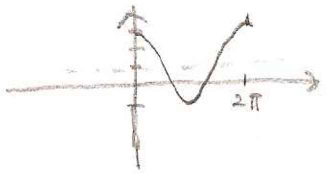
Problem 3 Find the area of the region between the inner loop and the outer loop of $r = 1 + 2\cos\theta$.

$\theta = \frac{2\pi}{3}$



$\theta = \frac{4\pi}{3}\left(-\frac{2\pi}{3}\right)$

$y = 1 + 2\cos x$



$1 + 2\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}\left(-\frac{2\pi}{3}\right)$

- Area of outer loop

$$= \frac{1}{2}\int_{-2\pi/3}^{2\pi/3} (1+2\cos\theta)^2\, d\theta$$

$$= 2\cdot\frac{1}{2}\int_0^{2\pi/3} (1+2\cos\theta)^2\, d\theta, \text{ using symmetry}$$

$$= \int_0^{2\pi/3} 1 + 4\cos\theta + 4\cos^2\theta\, d\theta$$

$$= \int_0^{2\pi/3} 1 + 4\cos\theta + 4\cdot\frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 2\pi + \frac{3\sqrt{3}}{2}$$

- Area of inner loop $= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1+2\cos\theta)^2\, d\theta$

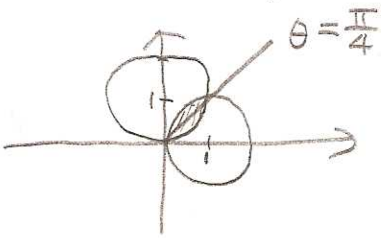
$$= \cdots = \pi - \frac{3\sqrt{3}}{2}$$

Hence, the area between 2 loops is $\left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \boxed{\pi + 3\sqrt{3}}$

Problem 4 Find the area of the region inside both of the polar curves $r = 2\sin\theta$ and $r = 2\cos\theta$.

Note that $r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta \Rightarrow x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$

Similarly, $r = 2\cos\theta \Rightarrow (x-1)^2 + y^2 = 1$



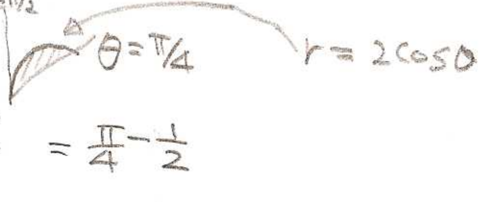
- points of intersection ; $2\sin\theta = 2\cos\theta$

$\Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \cancel{\frac{5\pi}{4}}$

- Area of lower half

$$= \frac{1}{2}\int_0^{\pi/4} (2\sin\theta)^2\, d\theta = \int_0^{\pi/4} 1 - \cos 2\theta\, d\theta = \frac{\pi}{4} - \frac{1}{2}$$

- Area of upper half



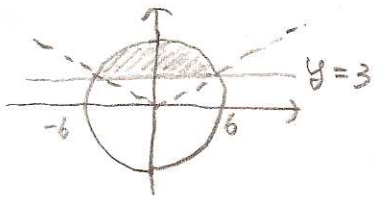
$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} (2\cos\theta)^2\, d\theta = \cdots = \frac{\pi}{4} - \frac{1}{2}$$

- Hence the area of the shaded region is

$$\left(\frac{\pi}{4} - \frac{1}{2}\right) + \left(\frac{\pi}{4} - \frac{1}{2}\right) = \boxed{\frac{\pi}{2} - 1}$$

Problem 5 Find the area of the curve inside the polar curve $r = 6$ and above the polar curve $r = 3\csc\theta$.

Note $r = 3\csc\theta \Rightarrow r\sin\theta = 3 \Rightarrow y = 3$



· Points of intersection: $6 = 3\csc\theta$

$\Rightarrow 2 = \csc\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence, the area is $\frac{1}{2}\int_{\pi/6}^{5\pi/6} r_{out}^2 - r_{in}^2 \, d\theta$

$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 6^2 - (3\csc\theta)^2 d\theta$

$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 36 - 9\csc^2\theta \, d\theta = 18\theta + \frac{9}{2}\cot\theta \Big|_{\theta=\pi/6}^{5\pi/6}$

$= \boxed{12\pi - 9\sqrt{3}}$

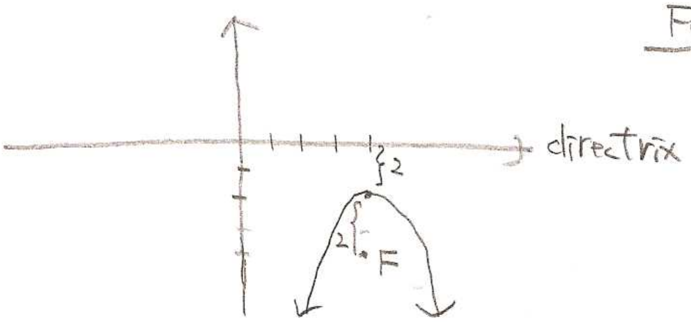
Problem 6 Sketch the conic section $(x-4)^2 = -2(y+2)$. Find the focus and directrix.

Only one variable is squared $\Rightarrow$ parabola

$4p = -2 \Rightarrow p = -\frac{1}{2}$

vertex $(4, -2)$

Focus $(4, -2-2) = (4, -4)$



directrix : $y = 0$

Problem 7 Sketch the conic section $\frac{(y-1)^2}{4} - (x+1)^2 = 1$. Find the foci, vertices and asymptotes.

hyperbola

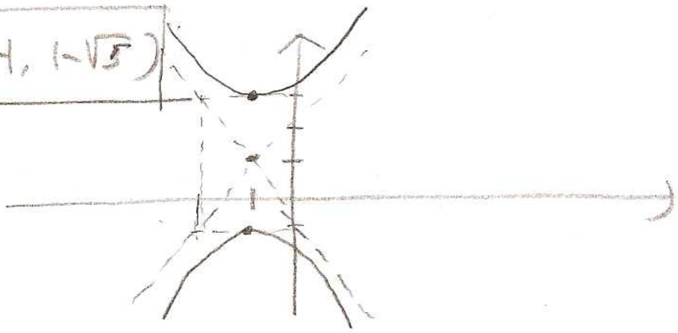
① center $(-1, 1)$

② $c = \sqrt{4+1} = \sqrt{5}$ = distance from center to foci

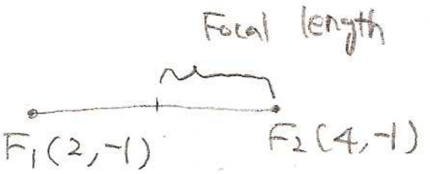
③ asymptotes : $(y-1) = \pm\sqrt{4}\,(x+1) \Rightarrow \boxed{y-1 = \pm 2(x+1)}$

④ opens upward / downward ⑤ vertices $(-1, 1+2) = (-1, 3)$, $(-1, 1-2) = (-1, -1)$

⑥ Foci $\boxed{(-1, 1+\sqrt{5}), (-1, 1-\sqrt{5})}$



Problem 8 Find the equation of the ellipse with foci $(2,-1)$ and $(4,-1)$, and the length of major axis 6.



center $\left(\frac{2+4}{2}, \frac{-1+(-1)}{2}\right) = (3, -1)$

Foci are on the horizontal line

$\Rightarrow 2a = 6 \Rightarrow a = 3$

$$\Rightarrow \frac{(x-3)^2}{3^2} + \frac{(y+1)^2}{b^2} = 1$$

Focal length $= c = 1$ and $c^2 = 3^2 - b^2$

$\Rightarrow 1^2 = 9 - b^2 \Rightarrow b^2 = 8$

Hence, $$\boxed{\frac{(x-3)^2}{9} + \frac{(y+1)^2}{8} = 1}$$

Problem 9 Evaluate each integral.

(a) $\int x\cos(2x)dx$

integration by parts $\begin{cases} u = x & du = dx \\ dv = \cos(2x)dx & v = \frac{1}{2}\sin(2x) \end{cases}$

$= \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)dx$

$= \boxed{\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$

(b) $\int \ln(x^2+1)dx$

integration by parts

$\begin{cases} u = \ln(x^2+1), & du = \frac{2x}{x^2+1}dx \\ dv = dx & v = x \end{cases}$

$= x\cdot\ln(x^2+1) - \int \frac{2x^2}{x^2+1}dx$

(Long division)

$= x\cdot\ln(x^2+1) - \int 2 - \frac{1}{x^2+1}dx$

$= \boxed{x\cdot\ln(x^2+1) - 2x + \tan^{-1}x + C}$

(c) $\int \frac{x^2}{x-1}dx$

Long Division

$= \int (x+1) + \frac{1}{x-1}dx$

$= \boxed{\frac{x^2}{2} + x + \ln|x-1| + C}$

(d) $\int \frac{3}{x^2-4}dx \quad = \int \frac{3}{(x+2)(x-2)}\,dx$

$$\frac{3}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} \Rightarrow 3 = A(x-2) + B(x+2)$$

$$\Rightarrow 3 = 4B, \quad 3 = -4A$$

$$\Rightarrow A = -3/4, \quad A = 3/4$$

$$\therefore \int \frac{3}{x^2-4}dx = -\frac{3}{4}\int \frac{1}{x+2}dx + \frac{3}{4}\int \frac{1}{x-2}dx$$

$$= \boxed{-\frac{3}{4}\ln|x+2| + \frac{3}{4}\ln|x-2| + C}$$

(e) $\int \frac{dx}{x^3+x}$

$$= \int \frac{dx}{x(x^2+1)},$$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow 1 = A(x^2+1) + Bx^2 + Cx$$

$$\Rightarrow 1 = (A+B)x^2 + Cx + A$$

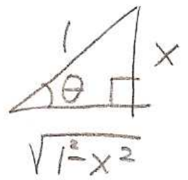
$$\Rightarrow A = 1, \quad B = -1, \quad C = 0$$

$$= \int \frac{1}{x} - \frac{x}{x^2+1}\,dx$$

$$= \boxed{\ln|x| - \frac{1}{2}\ln|x^2+1| + C}$$

(f) $\int \frac{x^2}{\sqrt{1-x^2}}dx$ (trigometric substitution ; $x = \sin\theta$

$dx = \cos\theta\, d\theta$



$$= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$$

$$= \int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$= \frac{1}{2}\theta - \frac{1}{4}\cdot 2\sin\theta\cdot\cos\theta + C$$

$$= \boxed{\frac{1}{2}\sin^{-1}x - \frac{1}{2}x\cdot\sqrt{1-x^2} + C.}$$

Problem 10 Find the improper integrals, or determine that they diverge.

(a) $\int_0^\infty xe^{-2x}dx$

$$= \lim_{b\to\infty} \int_0^b xe^{-2x}dx \qquad \begin{matrix} u = x & du = dx \\ dv = e^{-2x}dx & v = -\frac{1}{2}e^{-2x} \end{matrix}$$

$$= \lim_{b\to\infty} \left(-\frac{1}{2}xe^{-2x}\Big|_0^b + \frac{1}{2}\int_0^b e^{-2x}dx \right)$$

$$= \lim_{b\to\infty} \left(-\frac{1}{2}be^{-2b} - \frac{1}{4}e^{-2x}\Big|_0^b \right)$$

$$= \lim_{b\to\infty} -\frac{1}{2}\frac{b}{e^{2b}} - \frac{1}{4}e^{-2b} + \frac{1}{4} = \frac{1}{4}$$

$$\left(\because \lim_{b\to\infty}\frac{b}{e^{2b}} = \lim_{b\to\infty}\frac{1}{2e^{2b}} = 0, \quad \lim_{b\to\infty}\frac{1}{e^{2b}} = 0 \right)$$

(b) $\int_1^4 \frac{1}{(x-2)^2}dx$

$$= \int_1^2 \frac{1}{(x-2)^2}dx + \int_2^4 \frac{1}{(x-2)^2}dx$$

$$= \lim_{\substack{a\to 2\\ a<2}} \int_1^a \frac{1}{(x-2)^2}dx + \lim_{\substack{b\to 2\\ b>2}} \int_b^4 \frac{1}{(x-2)^2}dx$$

$$= \lim_{\substack{a\to 2\\ a<2}} -\frac{1}{x-2}\Big|_1^a + \lim_{\substack{b\to 2\\ b>2}} -\frac{1}{x-2}\Big|_b^4$$

$$= \underbrace{\lim_{\substack{a\to 2\\ a<2}} -\frac{1}{a-2}}_{\infty} + \left(\frac{1}{-1}\right) + \lim_{\substack{b\to 2\\ b>2}} -\frac{1}{2} + \frac{1}{b-2}$$

diverges.

Problem 11 Determine whether each series converges or diverges.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{10^{1/k}}$

$\lim_{k\to\infty} \frac{1}{10^{1/k}} = \frac{1}{10^0} = 1 \neq 0$

Hence diverges by the k-th term divergence test.

(b) $\sum_{k=4}^{\infty} \frac{(-1)^k \ln k}{k}$

- $a_k = \frac{\ln k}{k} \geq 0$ for all $k \geq 4$
- $\lim_{k\to\infty} \frac{\ln k}{k} = 0$ $\left(\because \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0\right)$
- $f(x) = \frac{\ln x}{x}$ is decreasing if $x \geq 4$ $\left(\because f'(x) = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2} \leq 0 \text{ for } x \geq 4\right)$

So converges by the alternating series test.

(c) $\sum_{k=1}^{\infty} \frac{\sqrt{k}+\sqrt[3]{k}}{k^2+k^3}$

Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$

$$\lim_{k\to\infty} \frac{\sqrt{k}+\sqrt[3]{k}}{k^3+k^2} \cdot \frac{k^3}{\sqrt{k}}$$

$$= \lim_{k\to\infty} \frac{k^3\sqrt{k} + k^3\sqrt[3]{k}}{k^3\sqrt{k} + k^2\sqrt{k}} = \lim_{k\to\infty} \frac{k^3\sqrt{k}}{k^3\sqrt{k}} = 1 \ (\neq 0, \infty)$$

and we know that $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ converges. $(p = \frac{5}{2} > 1)$

Hence the series converges.

(d) $\sum_{k=0}^{\infty} \frac{2^k}{3^k+1}$

Comparison Test with $\sum_{k=0}^{\infty} \frac{2^k}{3^k} = \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$;

$$\frac{2^k}{3^k+1} \leq \frac{2^k}{3^k} \quad \text{for all } k \geq 0.$$

We know that $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$ converges (geometric series with $r = \frac{2}{3} < 1$)

So the series converges.

(e) $\sum_{k=1}^{\infty} \frac{e^k}{k!}$

Ratio Test :

$$\lim_{k\to\infty} \frac{e^{k+1}}{(k+1)!} \cdot \frac{k!}{e^k} = \lim_{k\to\infty} \frac{e^k \cdot e^1}{k!(k+1)} \cdot \frac{k!}{e^k}$$

$$= \lim_{k\to\infty} \frac{e}{k+1} = 0. \quad (<1)$$

Hence the series converges

Problem 12 Find the interval of convergence of each power series.

(a) $\sum_{k=1}^{\infty} \frac{(2x-3)^k}{4^k}$

$$\lim_{k\to\infty} \left| \frac{(2x-3)^{k+1}}{4^{k+1}} \cdot \frac{4^k}{(2x-3)^k} \right| = \lim_{k\to\infty} \frac{1}{4} \cdot |2x-3|$$

$$= \frac{1}{4} \cdot |2x-3| < 1 \Rightarrow |2x-3| < 4$$

$$\Rightarrow -4 < 2x-3 < 4 \Rightarrow -1 < 2x < 7 \Rightarrow -\frac{1}{2} < x < \frac{7}{2}$$

At $x = -\frac{1}{2}$; $\sum_{k=1}^{\infty} \frac{(-4)^k}{4^k} = \sum_{k=1}^{\infty} (-1)^k$ diverges.

At $x = \frac{7}{2}$; $\sum_{k=1}^{\infty} \frac{4^k}{4^k} = \sum_{k=1}^{\infty} 1$ diverges

So, $\boxed{\left(-\frac{1}{2}, \frac{7}{2}\right)}$

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1} x^k$

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1} \cdot x^{k+1}}{4(k+1)^2-1} \cdot \frac{4k^2-1}{(-1)^k x^k} \right|$$

$$= \lim_{k\to\infty} \frac{4k^2-1}{4(k+1)^2-1} \cdot |x|$$

$$= \lim_{k\to\infty} \frac{4k^2-1}{4k^2+8k+3} \cdot |x| = |x| < 1 \Rightarrow -1 < x < 1$$

$x = -1$; $\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1} (-1)^k = \sum_{k=1}^{\infty} \frac{1}{4k^2-1}$ converges

(comparison test with $\sum_{k=1}^{\infty} \frac{1}{4k^2}$)

$x = 1$; $\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1} (1)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1}$ converges

(either AST or use absolute convergence)

Hence, $\boxed{[-1, 1]}$

Problem 13 Find the power series of the form $\sum_{k=0}^{\infty} a_k x^k$ converging to each function. In each case, specify the radius of convergence of the power series.

(a) $\frac{1}{1+x}$

$$= \frac{1}{1-(-x)}$$

Radius of convergence.

$$= \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k, \quad |-x| = |x| < 1$$

(b) $\ln(1+x)$ $= \int \frac{1}{1+x} dx + c = \int \sum_{k=0}^{\infty} (-1)^k x^k dx + c$

$$= \sum_{k=0}^{\infty} (-1)^k \int x^k dx + c = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} + c$$

$x = 0 \Rightarrow \ln 1 = c \Rightarrow c = 0$

$$\therefore \ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^k, \quad \text{radius of conv} = 1.$$

(c) $\frac{1}{1+x^2}$

$$= \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k},$$

$|-x^2| = |x^2| < 1 \Rightarrow |x| < 1 \Rightarrow$ radius of conv $= 1$

(d) $\arctan(x)$

$$= \int \frac{1}{1+x^2} dx + c = \sum_{k=0}^{\infty} (-1)^k \int x^{2k} dx + c$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + c$$

$x = 0 \Rightarrow \arctan 0 = c \Rightarrow c = 0$

$$\therefore \arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad \text{radius of conv} = 1$$

Problem 14 Use the first three nonzero terms of a Taylor series to approximate $\sin\left(\frac{1}{10}\right)$. Estimate the error in the approximation.

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\sin\left(\frac{1}{10}\right) = \frac{1}{10} - \frac{1}{3!}\left(\frac{1}{10}\right)^3 + \frac{1}{5!}\left(\frac{1}{10}\right)^5 - \frac{1}{7!}\left(\frac{1}{10}\right)^7 + \cdots$$

Alternating series

$$\approx \frac{1}{10} - \frac{1}{3!}\left(\frac{1}{10}\right)^3 + \frac{1}{5!}\left(\frac{1}{10}\right)^5$$

$|\text{Error}| < \frac{1}{7!}\left(\frac{1}{10}\right)^7$, using alternating series test error estimate.