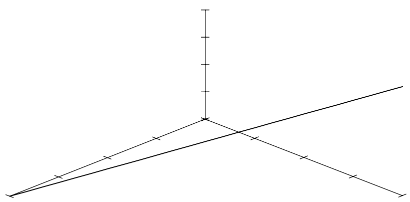
Section 12.5 Equations of Lines and Planes
Lines in space:
a. you need two points or
b. one point and a directional vector $\vec{d} = < d_1, d_2, d_3 >$



Directional vector : A vector parallel to the given line is called a directional vector.

The equation of the line with a directional vector $\vec{d} = < d_1, d_2, d_3 >$which goes through a point (x_0, y_0, z_0) is

Vector Equation of a line : $< x, y, z > = < x_0, y_0, z_0 > + t < d_1, d_2, d_3 >, \quad t \in \mathbb{R}$

Parametric Equation of a line : $x = x_0 + td_1,\ y = y_0 + td_2,\ \ z = z_0 + td_3, \quad t \in \mathbb{R}$

Symmetric Equation of a line : $\dfrac{x - x_0}{d_1} = \dfrac{y - y_0}{d_2} = \dfrac{z - z_0}{d_3} \quad (d_1 \neq 0, d_2 \neq 0, d_3 \neq 0)$

Example 1 Find the directional vector of the line.
a. $(x - 3,\ y + 2,\ z + 1) = t(3, 1, 0)$
b. $x = 2 - t,\ y = 3 + 2t,\ z = 5$
c. $\frac{x-3}{3} = \frac{y-2}{3} = \frac{z+5}{6}$

Example 2
a. Find a parametric equation of the line through $(1, 2, 3)$ and parallel to $\vec{d} = < 1, 0, -3 >$. And find two points on this line.

b. Fine a symmetric equation of the line which passes through $(1, 0, 2)$ and $(-2, -3, 5)$. Where is the line intersect yz-plane?

Example 3 True or False

a. If two lines in $\mathbb{R}^2$ have two different slopes then they must intersect.

b. If two lines in $\mathbb{R}^3$ have non-parallel directional vectors, then they must intersect.

Three cases of two different lines in $\mathbb{R}^3$: parallel, intersecting, skew(not parallel, no intersection)

Example 4 Let

$L_1 : x = 1 + 2t,\ y = t,\ z = 1 + 4t$

$L_2 : x = s,\ y = -2 + 2s,\ z = -2 + 3s$

Do they intersect? If so, find the point of intersection.

Note: If the equation of a line is not in parametric form, convert to parametric form when you find the point of intersection of two lines.

Planes:

Definition(Normal Vector): We say that a vector $\vec{n} = (n_1, n_2, n_3)$ is a normal vector to a plane P if $\vec{n} = (n_1, n_2, n_3)$ is perpendicular to the plane.

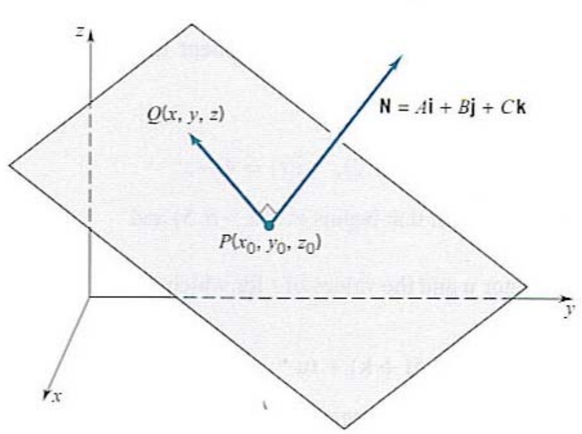
Plane in $\mathbb{R}^3$

a. You need 3 points or

b. a normal vector and a point to determine an equation of the plane.

Equation of the plane through (x_0, y_0, z_0) and with normal vector $\vec{n} = (a, b, c)$ is

$$(x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0,$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$



Example 5 Find the normal vector to the plane $x - 2y + 3z = 5$.

Example 6
a. Find the equation of the plane through $(-1,0,2)$ and with normal vector $\vec{n} = (1,2,3)$.

d. Find the equation of the plane passes through $(1,0,2),(3,4,5)$ and $(0,0,1)$.

a). We say that two planes are parallel if their normal vectors are parallel.
b) The angle between two planes is defined as the acute angle between their normal vectors.

Example 7 True or False
a. Two non-parallel plane must intersect.
b. Intersection between two non-parallel plane must be a line.
The following example tells us how to find the line of intersection of two planes.
Example 8 Let
$P_1 : x - 2y + 3z = 1$
$P_2 : x + 2y + 3z = 3$
a. Do they intersect?
b. Find two points on the intersection.

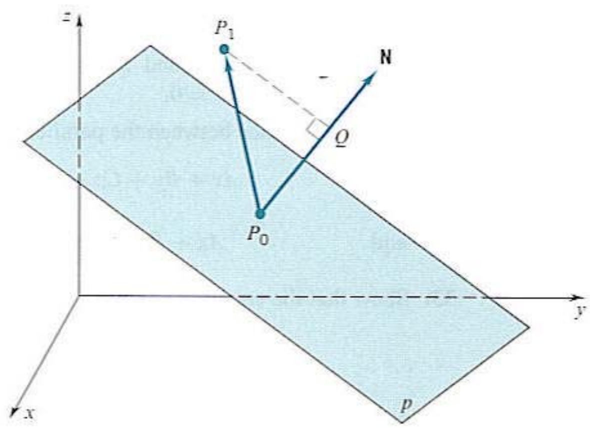
c. Find the equation of the line of intersection.

d. Compute $\vec{n}_1 \times \vec{n}_2$ of the cross produc of their normal vectors $\vec{n}_1$ and $\vec{n}_2$.

e. Is $\vec{d} \parallel \vec{n}_1 \times \vec{n}_2$?

Remark: a. As we see from the example 4, we can simply take $\vec{n}_1 \times \vec{n}_2$ as a directional vector of the line of intersection of two planes.
b. Note that the line of intersection is perpendicular to both of the normal vectors. So the directional vector is given by $\vec{n}_1 \times \vec{n}_2$.

Distance between a point to a plane



Let $P_1(x_1, y_1, z_1)$ and $P : ax + by + cz + d = 0$.
Let $P_0(x_0, y_0, z_0)$ be a point on the plane.
Then, $d(P_1, P) = \left|\text{comp}_{\vec{n}}\overrightarrow{P_0P_1}\right| = \frac{|\vec{n}\cdot\overrightarrow{P_0P_1}|}{|\vec{n}|} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Example 9 Find the distance between *yz*-plane and the point $(1, -3, -5)$.

HW: 3, 7, 13, 15, 17, 19, 25, 27, 31, 33, 47, 61.