Section 13.3 Arc Length and Curvature

Let $C : \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ be a differentiable curve.

- **the arc length** L from $t = a$ to $t = b$: $L = \int_a^b \left|\vec{r}'(t)\right| dt$

Example 1 Find the arc length of the curve $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ from $t = 0$ to $t = 2\pi$.

Parametrization of a curve with respect to arc length:

- **Arc length function** s : $s = s(t) = \int_a^t \left|\vec{r}'(u)\right| du$

Example 2 Reparametrize the helix $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ with respect to arc length from the point where $t = 0$ in the direction of increasing t.

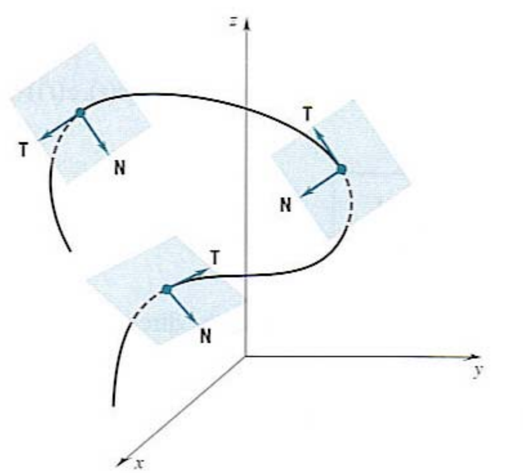
- **Curvature :** a measure how quickly the curve changes the direction
- **Curvature** κ : $\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|} = \frac{\left|\vec{r}'(t)\times\vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3}$, where $\vec{T}$ is the unit tangent vector.

Example 3 Find the curvature of $\vec{r}(t) =< \cos t,\ \sin t,\ t >$ by using $\frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|}$.

Curvature of a plane curve $y = f(x)$: $\kappa = \frac{|f''(x)|}{[1+(f(x))^2]^{\frac{3}{2}}}$

- the **unit tangent vector** $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$
- the **unit normal vector** $\vec{N}(t) = \frac{\vec{T}\,'(t)}{\left|\vec{T}\,'(t)\right|}$: an indication of the direction where the curve is turning at each point
- the **binormal vector** $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

Remark: $\vec{T}$, $\vec{N}$, $\vec{B}$ are mutually perpendicular unit vectors and $\vec{T} \times \vec{N} = \vec{B}$, $\vec{N} \times \vec{B} = \vec{T}$, $\vec{B} \times \vec{T} = \vec{N}$ (like the standard basis vectors $\vec{i}$, $\vec{j}$, $\vec{k}$)



- **osculating plane** at a point P : the plane containing $\vec{T}$ and $\vec{N}$ at P
- **normal plane** at a point P : the plane containing $\vec{N}$ and $\vec{B}$ at P

Example 4 Let $\vec{r}(t) =< 1,\ 2t,\ t^2 >$.

a. Find the unit tangent vector $\vec{T}$, unit normal vector $\vec{N}$, and binormal vector $\vec{B}$ at $t = 0$.

b. Find the osculating plane and the normal plane at $t = 0$.

HW: 1, 3, 9, 11, 15, 19, 33, 35
hint: #3: $e^{2t} + e^{-2t} + 2 = (e^t + e^{-t})^2$
For problems 19, 33, 35, you need to find t corresponging to the given point.
For example, in the problem 19, $\vec{r}(t) =< \sqrt{2}\,t,\ e^t,\ e^{-t} >=< 0, 1, 1 >$. So, $t = 0$.