Section 14.3 Partial derivatives

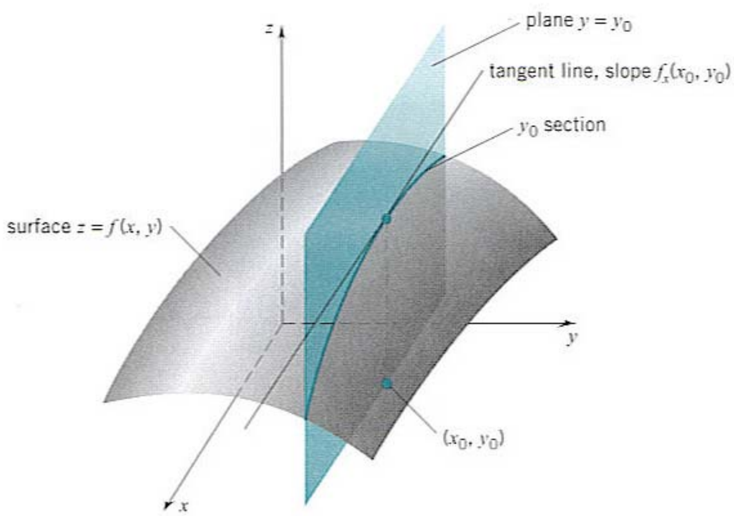
Let $z = f(x,y)$. Then, the **partial derivative** $f_x(x,y)$ and $f_y(x,y)$ are defined to be

- $f_x(x,y) = \lim_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$
- $f_y(x,y) = \lim_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h}$

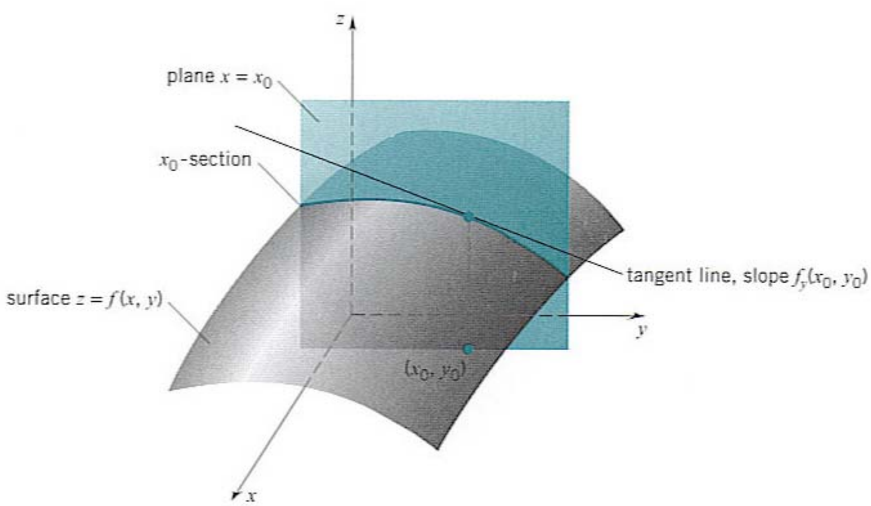
Notation

$f_x(x,y) = \dfrac{\partial z}{\partial x} = z_x \qquad f_y(x,y) = \dfrac{\partial z}{\partial y} = z_y$

$f_x(x_0,y_0)$ represents the **rate of change** of $f(x,y)$ along x-direction at (x_0,y_0)



$f_y(x_0,y_0)$ represents the **rate of change** of $f(x,y)$ along y-direction at (x_0,y_0)



Example 1 Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ by using the definition of partial derivative if $f(x,y) = x^2y$.

If you pay attention to the definition of the partial derivatives, then you will find the way to compute partial derivatives.

- **Treat y as a constant to find f_x. Treat x as a constant to find f_y.**

Example 2 Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

a. $f(x,y) = 3x^2 - xy + y$

b. $f(x,y) = x^2e^{-y}$

c. $f(x,y) = \tan^{-1}(xy)$

Example 3 Let $f(x,y) = \ln(x^2y)$. Find $f_x(1,2)$ and $f_y(1,2)$.

Example 4(Implicit differentiation)
Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by
$x^3 + y^3 + z^3 + 6xyz = 1.$

Example 5 Let $w = f(x,y,z)$. Write the definition of $\frac{\partial w}{\partial x}$, f_y and w_z.

Example 6 Find $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ for $w = x^2 e^{y/z}$

Mixed Partial Derivative and Higher Order Derivative Notations

$f_x = \frac{\partial f}{\partial x}$, $\qquad f_y = \frac{\partial f}{\partial y}$

$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}$, $\qquad (f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y}$

$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2}$, $\qquad (f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2}$

Example 7 Let $f(x,y) = x^2 y + xy^3$
a. Find f_{xy}, f_{yx}, f_{xx} and f_{yy}

b. Is $f_{xy} = f_{yx}$?

Example 8 Let $f(x,y) = \ln(x^2 + y^2)$. Find $\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$.

Theorem: $f_{xy}(a,b) = f_{yx}(a,b)$ if f_x and f_y exist and continuous on a disk D that contains (a,b).

HW: 13, 15, 19, 21, 23, 33, 41, 45, 49, 53, 59