## Section 14.5 The Chain Rule

### Chain Rule

Review: Let $y = f(x)$ and $x = x(t)$. Then, we can think $y$ as a function of $t$ and we can consider $\frac{dy}{dt}$.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

**Example 1** Let $y = e^{x^2+1}$ and $x(t) = \sin t + t + 1$. Find $\frac{dy}{dt}|_{t=0}$.

### Chain Rule in several variable

- **Case 1**: Let $z = f(x,y)$ and $x = x(t)$ and $y = y(t)$. Then, we can think $z$ as a function of $t$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

**Example 2** If $z = x^2 + y^2$, where $x = t$ and $y = t^2$, find $\frac{dz}{dt}$.

**Remark**: You can interprete $\frac{dz}{dt}$ in the example 2 as the rate of change of $z$ along the curve $\vec{r}(t) =< t, t^2 >$.

**Example 3** If $w = x^2 y + z\cos x$, where $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$, find $\frac{dw}{dt}$.

- **Case 2:** Let $z = f(x,y)$ and $x = x(s,t)$ and $y = y(s,t)$. Then, you can think $z$ as a function of $s$ and $t$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

**Remark**: See the tree diagram on page 920, 921 in the textbook.

**Example 4** Let $z = x^2 - 2xy + y^3$ and $x = s^2 \ln t$ and $y = 2st^3$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

**Example 5** Let $w = xy + yz + zx$ and $x = st, y = e^{st}$ and $z = t^2$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ when $s = 0,\ t = 1$.

**Example** (if time permits) Let $z = f(x,y)$ and $x = r^2 + s^2$ and $y = 2rs$. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial r^2}$.

- **Explicit and Implicit function**

Explicit function is of the form $z = f(x,y)$.
Implicit function is of the form $g(x,y,z) = c$ (Relation is given by an equation)

- **Implicit differentiation:**

Let $F(x,y) = 0$ defines $y$ implicitly as a differentiable function of $x$. Then,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

**Example 6** Find $\frac{dy}{dx}$ if $\cos(x-y) = xe^y$.

Let $F = F(x,y,z)$ is continuously differentiable, and $z = f(x,y)$ is given implicitly by $F(x,y,z) = 0$, then at the points $(x,y,z)$ where $\frac{\partial u}{\partial z} \neq 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

**Example 7** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

a. $z^3 + xz + y^2 + xy = 2$

b. $\cos xyz + \ln(x^2 + y^2 + z^2) = 0$

HW: 3, 7, 11, 15, 19, 21, 25, 29, 31, 43
hint: 43: Let $u = x - y$. Then $\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}$.