Section 14.6 Directional Derivatives and the Gradient Vector

- **The Gradient Vector**

Let $z = f(x,y)$. Then, the gradient $\vec{\nabla} f$ of f is

$$\vec{\nabla} f = < f_x, f_y >$$

Let $w = f(x,y,z)$. Then, the gradient $\vec{\nabla} f$ of f is

$$\vec{\nabla} f = < f_x, f_y, f_z >$$

Example 1 Let $f(x,y) = x^2 + y^2 + 2xy$. Compute $\vec{\nabla} f$.

- **Directional Derivatives**

The directional derivatives of f at (x_0,y_0) in the direction of a **unit vector** $\vec{u} = < a,b >$ is

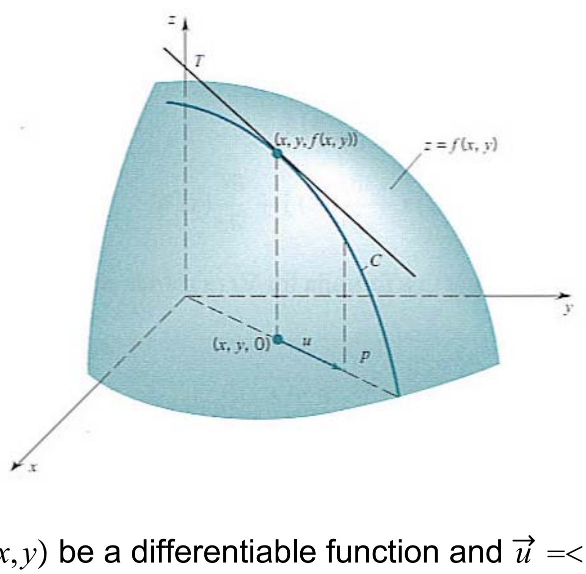
$$D_{\vec{u}} f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0,y_0)}{h}.$$

Note that $D_{\vec{i}} f = f_x$, and $D_{\vec{j}} f = f_y$.

f_x : rate of change of f along the $x-axis$

f_y : rate of change of f along the $y-axis$

$D_{\vec{u}} f$: rate of change of f along a unit vector $\vec{u}$



- Let $f(x,y)$ be a differentiable function and $\vec{u} = < a,b >$ be a unit vector. Then,

$$D_{\vec{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}.$$

Example 2 Find the directional derivative of the function $f(x,y) = x\cos y$ at $(1,\pi)$ in the direction of $2\vec{i} - \vec{j}$.

- **Maximizing the Directional Derivative**

Suppose f is differentiable functinon of two or three variables. Then

- the maximum value of $D_{\vec{u}}f$ is $\left|\vec{\nabla}f\right|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\vec{\nabla}f$.
- the mimimum value of $D_{\vec{u}}f$ is $-\left|\vec{\nabla}f\right|$ and it occurs when $\vec{u}$ has the opposite direction to the gradient vector $\vec{\nabla}f$.

f **increases most rapidly along the direction of** $\vec{\nabla}f$ **and decreases most rapidly along the direction of** $-\vec{\nabla}f$**.**

Example 3 Suppose that the temperature at each point of a metal plate is given by the function $T(x,y) = e^x \cos y + e^y \cos x$

a. In what direction does the temperature increase fatest at the point $(0,0)$? What is the maximum rate of increase?

b. In what direction does the temperature decrease most rapidly at the point $(0,0)$?

- **Tangent Planes to Level Surfaces**

Suppose S is a surface given by $F(x,y,z) = k$ and C be any curve on the surface S which passses throuh the point $P(x_0,y_0,z_0)$.

Note that the curve C is described by a vector function $\vec{r}(t) =< x(t),y(t),z(t) >$.

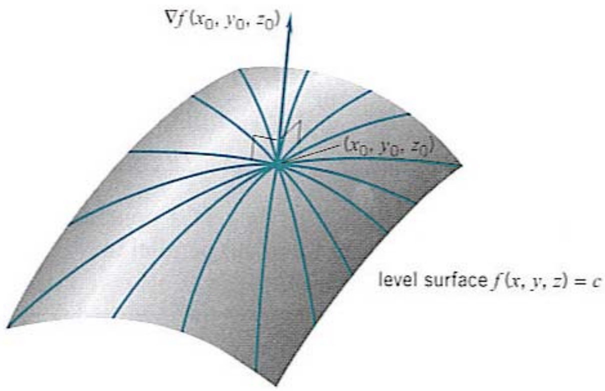
Then

$$\vec{\nabla}F \cdot \vec{r}\,'(t) = 0.$$

So the gradient vector $\vec{\nabla}F$ **is perpendicular to the tangent vector** $\vec{r}\,'$ **at any values of** t if $\vec{\nabla}F(t) \neq \vec{0}$.

- Hence **the equation of the tangent plane to the surface** $F(x,y,z) = k$ **at** $P(x_0,y_0,z_0)$ is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$$

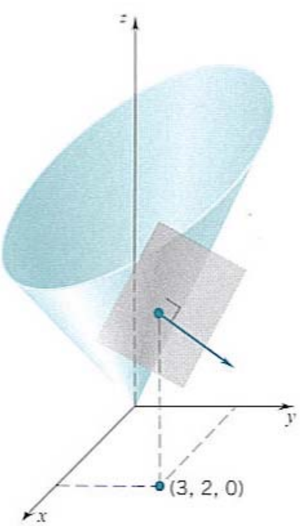


The gradient vector $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level surface at (x_0, y_0, z_0)

- **The normal line to S at $P(x_0,y_0,z_0)$** is the line passing through $P(x_0,y_0,z_0)$ and parallel to the gradient vector (perpendicular to the tangent plane): the symmetirc equation is given by

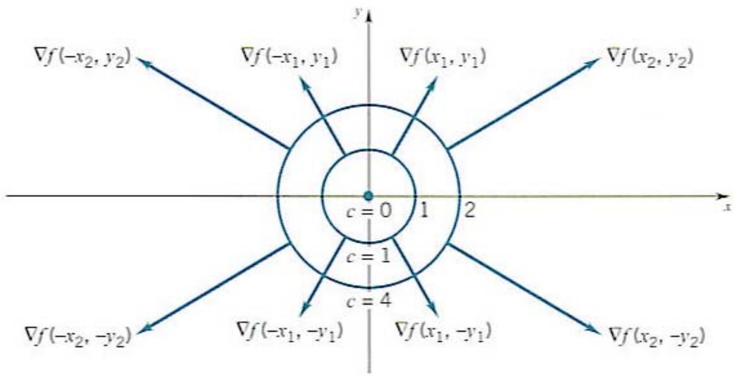
$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}.$$

Example 4 Find an equation for the tangent plane and normal line to the elliptic cone $x^2+4y^2=z^2$ at the point $(3,2,5)$.



Remark: Let C be a curve given by $f(x,y)=k$. Then by the similar way, the gradient vector $\vec{\nabla} f$ is perpendicular to the tangent vector $\vec{r}\,'$ to the curve C given by $f(x,y)=k$.
So **the equation of the tangent line to the curve** $C: f(x,y)=k$ **is**

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=0.$$



HW: 3, 7, 9, 11, 15, 19, 23, 25, 27, 29, 31, 39, 41, 45

Hint: #29: T is inversely proportional to d means $T = \frac{k}{d}$ (k: constant).