

Section 14.8 Lagrange Multipliers

We consider the following problems:

Find the maximum (minimum) values of $f(x, y)$ subject to the constraint $g(x, y) = k$.

Find the maximum (minimum) values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

Geometric interpretation of the problems:

Find extreme values of $f(x, y)$ when (x, y) is on the curve $C : g(x, y) = k$.

Find extreme values of $f(x, y, z)$ when (x, y, z) is on the surface $\Gamma : g(x, y, z) = k$.

Idea behind Lagrange multiplier method: At the extreme values along the curve C (or a surface Γ), **gradient f and g are parallel to each other**. So, $\vec{\nabla}f = \lambda \vec{\nabla}g$ **for some scalar λ at the points of extreme values**.

More precisely,

If f has an extreme value at (a, b) when (a, b) is on the curve $C : g(x, y) = k$, then $\vec{\nabla}f \perp C$ at (a, b) , i.e., gradient of f is perpendicular to the tangent vector of the curve C at (a, b) .

If f has an extreme value at (a, b, c) when (a, b, c) is on the surface $\Gamma : g(x, y, z) = k$, then $\vec{\nabla}f \perp C$ at (a, b, c) for the curves C that lie on the surface Γ .

We also know that $\vec{\nabla}g \perp C$ in either cases.

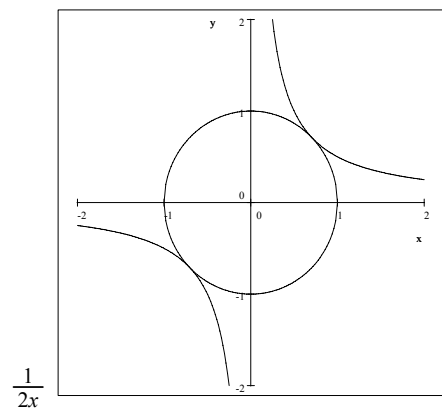
Hence, we reach the conclusion that $\vec{\nabla}f \parallel \vec{\nabla}g$ at (a, b) (or (a, b, c)).

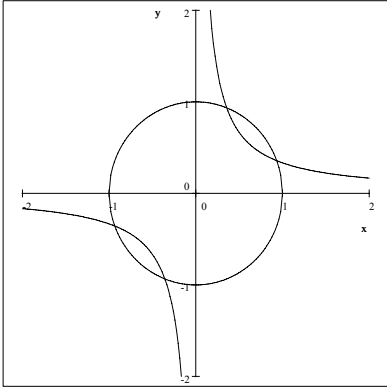
So, $\vec{\nabla}f = \lambda \vec{\nabla}g$ at (a, b) (or (a, b, c)) and λ is called a Lagrange multiplier.

Method of Lagrange Multipliers: this method is used to find the maximum (minimum) values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:

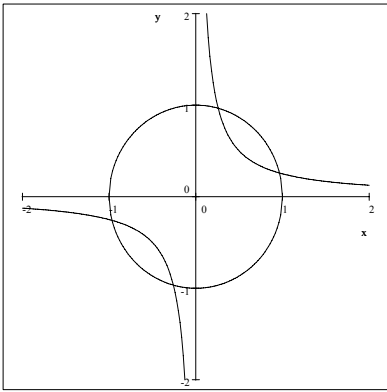
1. Find all values of x, y, z such that $\vec{\nabla}f(x, y, z) = \lambda \vec{\nabla}g(x, y, z)$ **and** $g(x, y, z) = k$.
2. Evaluate f at all the points (x, y, z) from step 1. The largest is the maximum value of f ; the smallest is the minimum value of f .

Example 1 Find the maximum and minimum of $f(x, y) = xy$ on the unit circle $x^2 + y^2 = 1$.





$$\frac{1}{3x}$$



$$\frac{1}{4x}$$

Example 2 A rectangular box without a top is to be made from 12 m^2 of cardboard. Find maximum volume of such a box..

Example 3 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farther from the point $(3, 1, -1)$.

Example 4 Find the extreme values of $f(x, y) = xy$ on the disk $x^2 + y^2 \leq 1$.

I will cover the method of Lagrange multipliers with two constraints if time permits.

Two Constraints

Consider the following problem: find the extreme values of $f(x, y, z)$ with two side conditions $g(x, y, z) = k$ and $h(x, y, z) = c$.

Geometric meaning: find the extreme values of f when (x, y, z) lies on the curve of intersection of $g(x, y, z) = k$ and $h(x, y, z) = c$.

Method of Lagrange multipliers:

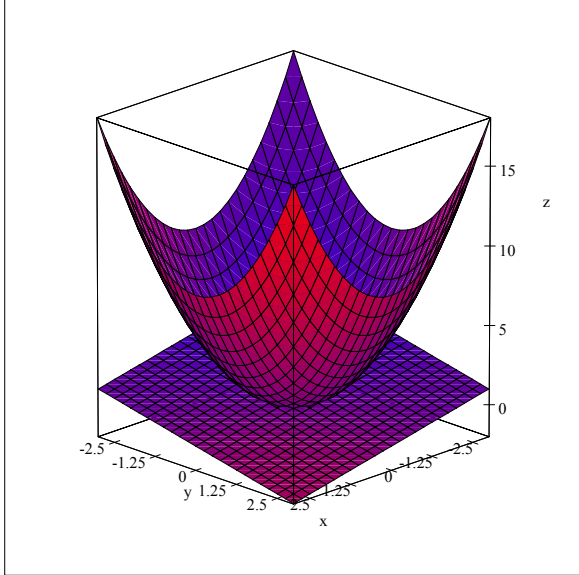
1. Find all values of x, y, z such that $\vec{\nabla}f(x, y, z) = \lambda\vec{\nabla}g(x, y, z) + \mu\vec{\nabla}h(x, y, z)$, $g(x, y, z) = k$, and $h(x, y, z) = c$.
2. Evaluate f at all the points (x, y, z) from step 1. The largest is the maximum value of f ; the smallest is the minimum value of f .

The equations obtained from step 1 are

$$\begin{aligned}f_x &= \lambda g_x + \mu h_x, & f_y &= \lambda g_y + \mu h_y, & f_z &= \lambda g_z + \mu h_z, \\g(x, y, z) &= k, & \text{and } h(x, y, z) &= c.\end{aligned}$$

Example 5 The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are highest and lowest.

Set up the problem with two side conditions: find maximum and minimum of $f(x, y, z) = z$ with constraints $g(x, y, z) = x + y + 2z = 2$ and $h(x, y, z) = x^2 + y^2 - z = 0$.



HW: 5, 7, 9, 19, 27, 37, 39(if we cover example 5)