Section 15.1 & 15.2 Double Integrals over Rectangles and Iterated Integrals

$\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy = \int_c^d \int_a^b f(x, y)\, dx\, dy$ (or $\int_a^b \int_c^d f(x, y)\, dy\, dx$) is called an *Itegrated integral* and is evaluated by integrating with respect to x (keeping y constant) and then with respect to y (similarly with respect to y and then x).

Example 1 Evaluate the iterated integrals.

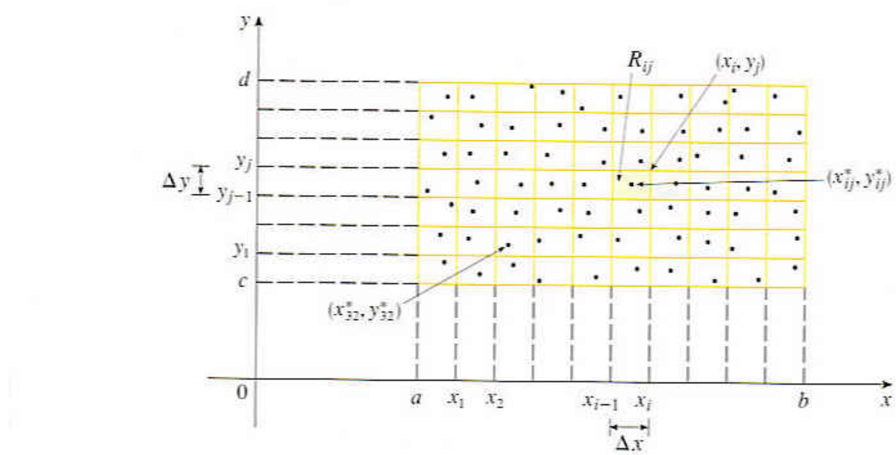
a. $\int_0^3 \int_1^2 (x^4 - 2y)dxdy$

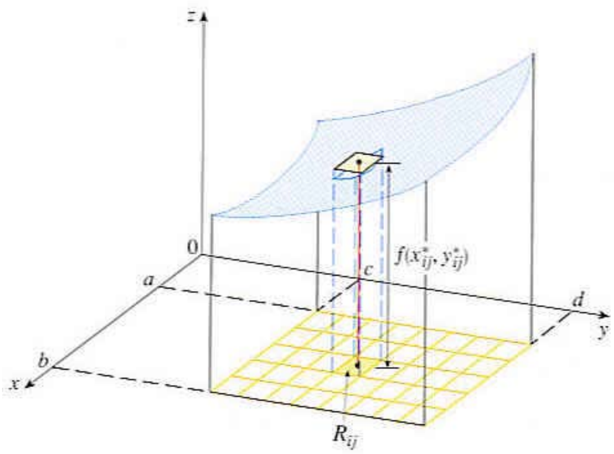
b. $\int_1^2 \int_0^3 (x^4 - 2y)dydx$

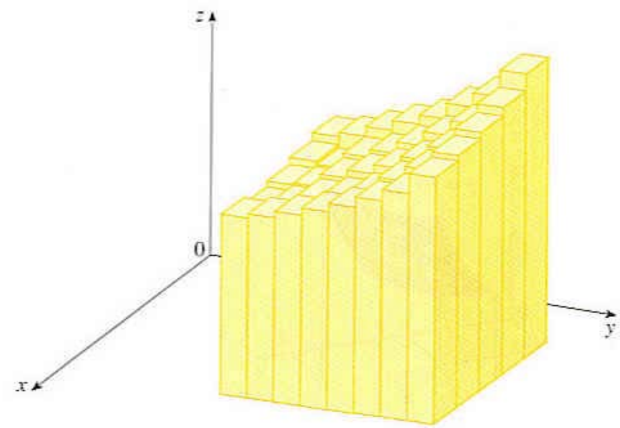
The double integral $\iint_R f(x,y)dA$ over the rectangle R is defined as the limit of double Riemann sum (see page 969 in the textbook).

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.







$f(x_{ij}^*, y_{ij}^*)\,\Delta A$ $\qquad$ $V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$

Here, dA is *the element of area* and $dA = dx\,dy$ (in Cartesian coodrinates).
We evaluate the double integrals by iterated integrals.
Fubini's Theorem: If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

More generally, this is true if f is bounded on R , f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.
So, the order of integration does not matter. But it might happen that integrating with respect to one order is easier than integrating with respect to another.
Example 2 Evaluate $\iint_R y\sin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

Volume and Area by double integration

Let $z = f(x,\ y)$ be continuous on the rectangle R in the xy-plane. Then,

- $\int\int_R f(x,y)\ dx\ dy$ =Volume of the solid that lies under the surface $z = f(x,\ y)$ and above the rectangle R if $f(x,y) > 0$ on R.

(In general, $\int\int_R f(x,y)\ dx\ dy$= (Volume above xy-plane)− (Volume below the xy-plane))

- $\int\int_R 1\ dx\ dy$ =Area of the region R

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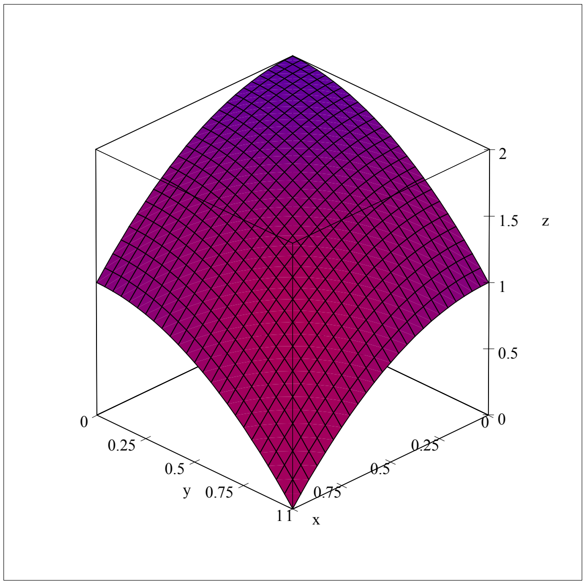
Example 3 Find the volume of the solid S that is under the surface $z = 2 - x^2 - y^2$ and above the rectangle $R : 0 \le x \le 1,\ \ 0 \le y \le 1$.



Average Value
The average value of a function $f(x, y)$ on a rectangle R is defined as

$$f_{\text{avg}} = \frac{1}{\text{A}(R)} \iint_R f(x, y)\, dA,$$

where $\text{A}(R)$ is the area of R.

Example 4 Let $f(x,y) = x\sin(xy)$. Find the average value of f over the reatangle $[0, \frac{\pi}{2}] \times [0, 1]$.

HW: **Section 15.2** : 3, 5, 11, 13, 15, 17, 23, 33
Hints:
11: $e^{2x-y} = e^{2x}e^{-y}$
17: Use integration by parts $\int_a^b u dv = uv|_a^b - \int_a^b v du$.