Section 15.3 Double Integrals over General Regions

We evaluate the double integrals over the general retion D by itegrated integrals.

- Suppose that the region D is expressed as

$$D = \{(x,y) \mid a \le x \le b,\ \ g_1(x) \le y \le g_2(x)\ \}.$$

That means D is the region bounded below and above by $y = g_1(x)$ and $y = g_2(x)$ and to the left and right by $x = a$ and $x = b$.
Then,

$$\iint_D f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\ dx.$$

- Similarly, if the region D is expressed as

$$D = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ \ c \le y \le d\ \},$$

then

$$\iint_D f(x,y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\ dy.$$

Example 1 Let D be the region bounded by $y = x^2$, $y = 2 - x$.
a. Sketch the region D.

b. Evaluate $\int\int_D (x + 2y\ )\ dA$.

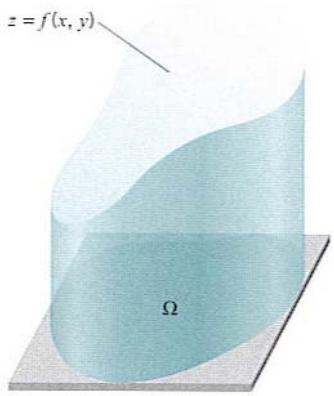
Volume and Area by double integration over general region D

Let $z = f(x, y)$ be continuous on the region D in the xy-plane.

Consider a solid S described by $\{(x, y, z) \mid (x, y) \in D \text{ and } 0 \le z \le f(x,y)\}$.

Then,

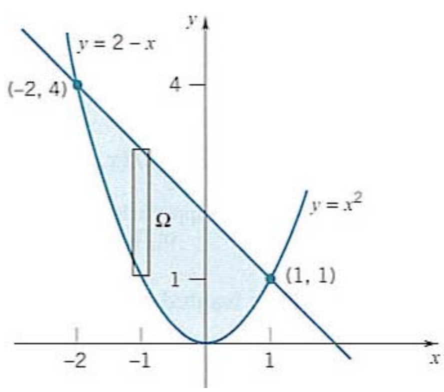
- $\iint_D f(x,y)\, dA$ =Volume of the solid S that lies under the surface $z = f(x, y)$ and above the region D if $f(x,y) \ge 0$ on D.

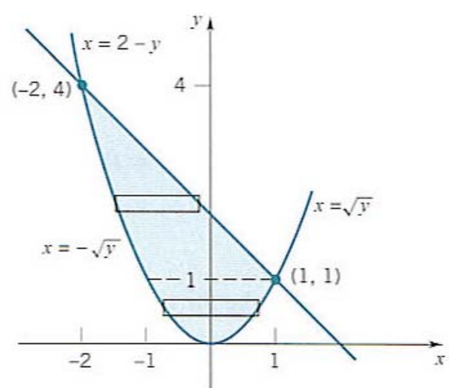


In general, $\iint_D f(x,y)\, dA$ = (Volume above xy-plane)– (Volume below the xy-plane).

- $\iint_D 1\, dA$ =Area of the region D.

Example 2 Let D be the region bounded by $y = x^2$, $y = 2 - x$. Express the area of region D in two different ways.





Example 3 Let D be the region bounded by $y = x^2$ and $y = \sqrt{x}$.
Set up an iterated integral for the volume of the solid that is under $z = x + y + 1$ and above the region D.

Example 4 Find the voulme of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 0$, $y = 0$ and $z = 0$.

Example 5 Evaluate the integrals by changing the order of integration.

a. $\int_0^1 \int_x^1 \sin(y^2)dy$

b. $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dxdy.$

Properties of Double Integrals: All the typical properties of the ingetral hold for the double integral. For example, constants can be pulled out and double integral of sum of two functions is sum of double integrals of each function

- $\iint_D (f+g)\, dA = \iint_D f\, dA + \iint_D g\, dA$
- $\iint_D cf\, dA = c\iint_D f(x, y) dA$
- If $f \geq g$ for all (x,y) in D, then $\iint_D f\, dA \geq \iint_D g\, dA$.
- $\iint_{D_1 \cup D_2} f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA$.
- If $m \leq f(x, y) \leq M$ for all (x, y) in D, then $mA(D) \leq \iint_D f\, dA \leq MA(D)$

HW: 7, 13, 15, 19, 25, 35, 39, 41