15.4 Double Integrals in Polar Coordinates

Polar Coordinates:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ r^2 = x^2 + y^2,\ \ \tan\theta = \frac{y}{x}.$$

Change to Polar Coordinates in Double Integral:

Let D be a polar region of the form

$$D = \{(r,\ \theta)\mid \alpha \le \theta \le \beta,\ a \le r \le b\}.$$

Then,

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta$$

Let D be a polar region of the form

$$D = \{(r,\ \theta)\mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\ \}.$$

Then,

$$\iint_D f(x,\ y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta.$$

Note that $dA = dxdy = dydx = rdrd\theta$.

Do not forget r when you conver to polar coordinates.

Example 1

a) The disk $x^2 + y^2 \le a^2$ with radius a, center $(0,0)$:

In polar coordinates, $0 \le r \le a,\ 0 \le \theta \le 2\pi$.

b) The portion of the disk $x^2 + y^2 \le a^2$ in the upper half plane:

In ploar coordinates, $0 \le r \le a,\ 0 \le \theta \le \pi$.

c) The ring $a^2 \le x^2 + y^2 \le b^2$:

In polar coordinates, $a \le r \le b,\ 0 \le \theta \le 2\pi$.

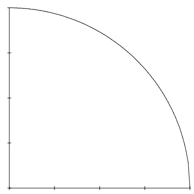
d) The portion of the ring $a^2 \le x^2 + y^2 \le b^2$ in the first quadrant:

In polar coordinates, $a \le r \le b,\ 0 \le \theta \le \frac{\pi}{4}$.

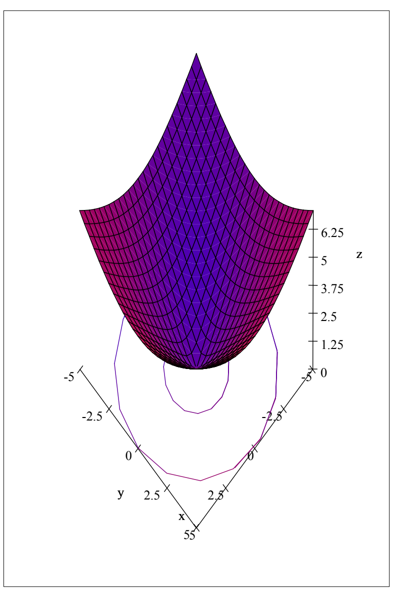
e) The disk $(x-a)^2 + y^2 \le a^2$ with radius a, center $(a,0)$:

In polar coordinates, $0 \le r \le 2a\cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

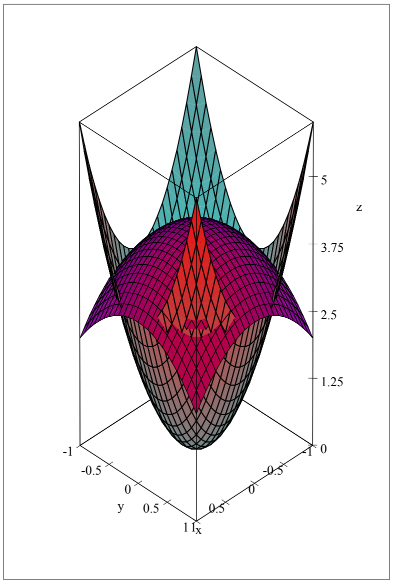
Example 2 Evaluate $\iint_R x\, dA,$ where R is the portion of the disk $x^2 + y^2 \le 16$ that lies in the first quadrant.



Example 3 Use polar coordinates to find the volume of the solid under $z = \sqrt{x^2 + y^2}$ and above the ring $4 \le x^2 + y^2 \le 25$ (set up only).



Example 4 Use polar coordinates to find the voulume of the region bounded by the paraboloid $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$ (set up only).



Example 5 Convert the integral $\int_{-a}^{a}\int_{0}^{\sqrt{a-y^2}} (x^2+y^2)^{\frac{3}{2}}\, dxdy$ to polar coordinates.

Example 6 Use double integral to find the area of one loop of the four leaved rose $r = \cos 2\theta$.

HW: 7, 9, 13, 15, 19, 23, 27
hint: #15: use a double angle formula $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ to evaluate the ingetral.