15.7 Triple Integrals

Triplr integral over the rectangular box is of the form $\int\int\int_B f(x, y, z)\ dV$. We say that dV is a *volume elemenet* (or an element of volume).
Similar to the double integral, we use iterated integrals to evaluate them.
Fubini's Theorem for Triple Integrals: If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\ dxdydz.$$

The iterated integrals on the right side means we integrated with respect to x first (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally integrate with respect to z. We can also change the order of integration, for example, $dxdzdy$, etc, and all the different orders of integration give the same value.

Example 1 Evaluate the triple integral $\int\int\int_B xyz^2\ dV$, where $B = [0, 1] \times [-1, 2] \times [0, 3]$.

Now, consider the triple integral over a general bounded region E in three dimensional space. We write the triple integrals over a solid E as $\int\int\int_E f(x, y, z)dV$ and evaluate them by iterated integrals.

Reduction to Iterated Integrals:

- Suppose that the region E is expressed as

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\},$$

where D is the projection of E onto xy-plane.
Then

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA.$$

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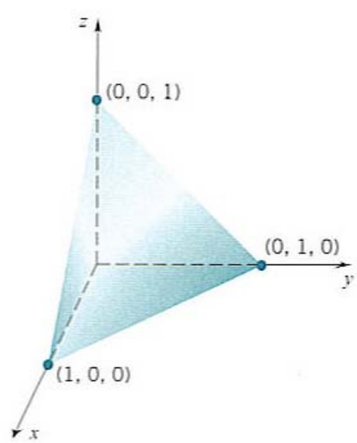
Example 2 Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by four planes $x + y + z = 1$, $x = 0$, $y = 0$, and $z = 0$.



Example 3 Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$ where E is bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Example 4 Set up a iterated integrals for $\iiint_E dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $y + z = 1$, and $x + z = 1$.

Applications of Triple Integrals

Volume $V(E)$ of a solid E : $V(E) = \iiint_E dV$

Mass m : let $\rho(x, y, z)$ be a density function of a solid object which occupies the region E, then the total mass is $m = \iiint_E \rho(x, y, z)\, dV$.

Moments about yz-plane, xz-plane and xy-plane are respectively

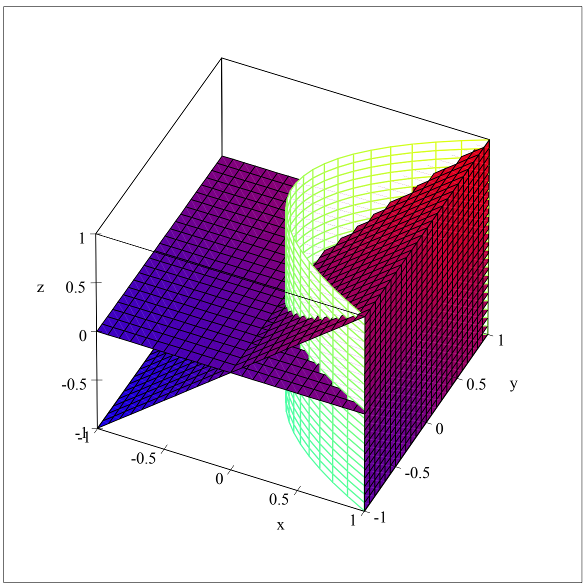
$M_{yz} = \iiint_E x\rho(x, y, z)\, dV$, $\quad M_{xz} = \iiint_E y\rho(x, y, z)\, dV$, and $M_{xy} = \iiint_E z\rho(x, y, z)\, dV$.

Center of Mass $(\overline{x}, \overline{y}, \overline{z})$:

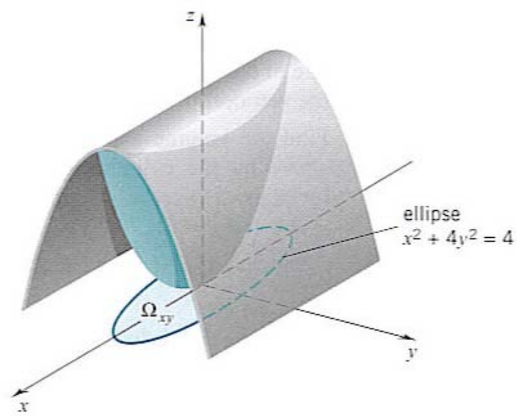
$\overline{x} = \frac{M_{yx}}{m}, \quad \overline{y} = \frac{M_{xz}}{m}, \quad \overline{z} = \frac{M_{xy}}{m}$

There are more applications (see the textbook: page 1014).

Example 5 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$, $x = z$, $z = 0$ and $x = 1$.



Example 6 Use triple integration to find the volume of the solid T bounded above by the $z = 4 - y^2$ and bounded below by $z = x^2 + 3y^2$. (set up only)



HW: 3, 7, 11, 13, 15, 17, 23, 37a), b) only