

## Section 15.9 Change of Variables in Multiple Integrals

Recall a change of variable in a single variable:

$$\int_a^b f(x) dx = \int_c^d f(x(u)) \frac{dx}{du} du,$$

where  $a = x(c)$ ,  $b = x(d)$ .

- **Jacobian(determinant):**

Let  $x = x(u, v)$ ,  $y = y(u, v)$ . Then Jacobian is defined as

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Let  $x = x(u, v, w)$ ,  $y = y(u, v, w)$ , and  $z = z(u, v, w)$ . Then Jacobian is defined as

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

**Example 1** Let  $x = u^2 - v^2$ ,  $y = 2uv$ . Find the Jacobian.

- **Change of Variables in double integrals:**

$$\iint_{R_{xy}} f(x, y) dA = \iint_{R_{uv}} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

So,  $dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$ .

Note that the **absolute value** of the Jacobian is used in the change of variables formula.

**Example 2** (Polar coordinates revisited): Let  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Write down the change of variables formula in double integrals.

Miscellaneous fact:  $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$ .

When is the change variables useful?

- 1) If the region of integration  $R_{xy}$  is awkward, choose a suitable transformation so that the corresponding region  $R_{uv}$  in the  $uv$ -plane have a convenient description.
- 2) If  $f(x,y)$  is difficult to integrate, then the form of  $f(x,y)$  may suggest a transformation.

Note that the limits of integration for the  $uv$  -plane are determined from the figure and are not directly related to the limits that would be assigned by the corresponding iterated integral in the  $xy$  -plane.

**Example 3** Evaluate  $\iint_R (x + 2y)dA$ , where  $R$  is the parallelogram bounded by the lines  $x + y = 1$ ,  $x + y = 4$ ,  $x - 2y = -2$ , and  $x - 2y = 1$ .

**Example 4** Evaluate  $\iint_R e^{\frac{x+y}{x-y}} dA$ , where  $R$  is the trapezoidal region with vertices  $(1, 0)$ ,  $(2, 0)$ ,  $(0, -2)$  and  $(0, -1)$ .

● **Change of Variables in Triple Integrals:**

$$\iiint_{R_{xyz}} f(x, y, z) dV = \iiint_{R_{uvw}} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

**Example 5** (Spherical coordinates revisited): Let  $x = \rho \sin \phi \cos \theta$ ,  $y = \rho \sin \phi \sin \theta$ ,  
 $z = \rho \cos \phi$ .

Then  $\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = -\rho^2 \sin \phi$ .

HW: 1, 3, 5, 11, 13, 17a), 19