# MATH 242C EXAM 1

**NAME: Solution (White)**

**Problem 1** Let $\vec{a} =< -1, 3, 4 >$ and $\vec{b} =< 1, 2, -3 >$.
a)(3 pt) Find the unit vector in the direction of $\vec{a}$.
solution)

$$\frac{\vec{a}}{|\vec{a}|} = \frac{< -1, 3, 4 >}{\sqrt{26}}.$$

b)(4 pt) Find the vector projection of $\vec{b}$ onto $\vec{a}$.

$$\text{Proj}_{\vec{a}}\vec{b} = \left(\vec{b} \cdot \frac{\vec{a}}{|\vec{a}|}\right) \frac{\vec{a}}{|\vec{a}|} = \frac{-7}{\sqrt{26}} \frac{< -1, 3, 4 >}{\sqrt{26}} = \frac{-7}{26} < -1, 3, 4 > .$$

**Problem 2**(5 pt) Find $\vec{r}(t)$ given that $\vec{r}\,'(t) = t\vec{i} - \vec{j} + e^{2t}\vec{k}$ and $\vec{r}(0) = \vec{i} - \frac{1}{4}\vec{j} + \vec{k}$.
solution)

$$\vec{r}(t) = \int t dt\ \vec{i} - \int 1 dt \vec{j} + \int e^{2t} dt \vec{k}.$$

$$\vec{r}(t) = \left(\frac{t^2}{2} + c_1\right)\vec{i} - (t + c_2)\,\vec{j} + \left(\frac{1}{2}e^{2t} + c_3\right)\vec{k}.$$

$$\vec{r}(0) = c_1\vec{i} + c_2\vec{j} + \left(\frac{1}{2} + c_3\right)\vec{k} = \vec{i} - \frac{1}{4}\vec{j} + \vec{k}.$$

$$c_1 = 1, \quad c_2 = \frac{1}{4}, \quad c_3 = \frac{1}{2}.$$

$$\vec{r}(t) = \left(\frac{t^2}{2} + 1\right)\vec{i} - \left(t + \frac{1}{4}\right)\vec{j} + \left(\frac{1}{2}e^{2t} + \frac{1}{2}\right)\vec{k}.$$

**Problem 3**(5 pt) Find the parametric equation of the line which passes through $(1, 4, 6)$ and $(2, -1, 3)$.

solution) A vector parallel to the line is

$$< 2 - 1, -1 - 4, 3 - 6 >= (1, -5, -3 > .$$

The parametric equation of the line is

$$x = 1 + t, \quad y = 4 - 5t, \quad z = 6 - 3t.$$

Other possible answers:

$$x = 1 - t, \quad y = 4 + 5t, \quad z = 6 + 3t,$$

$$x = 2 + t, \quad y = -1 - 5t, \quad z = 3 - 3t.$$

$$x = 2 - t, \quad y = -1 + 5t, \quad z = 3 + 3t.$$

**Problem 4**(9 pt) Find the equation of the plane that contains $(1, 1, 1)$, $(2, 4, 3)$ and $(-1, -2, -1)$. Simplify your answer as much as possible. solution) Let $P(1, 1, 1)$, $Q(2, 4, 3)$, and $R(-1, -2, -1)$.

$$\vec{PQ} =< 1, 3, 2 >, \quad \vec{PR} =< -2, -3, -2 > .$$

Unit vector to the plane is

$$\vec{PQ} \times \vec{PR} = -2\vec{j} + 3\vec{k}.$$

The equation of the plane is

$$-2(y - 1) + 3(z - 1) = 0.$$

$$2y - 3z = -1.$$

**Problem 5** Let $\vec{r}(t) = 3\sin t\vec{i} + 4t\vec{j} + 3\cos t\vec{k}$.
a) (5 pt) Find the unit tangent vector.
solution)

$$\vec{r}\,'(t) = 3\cos t\vec{i} + 4\vec{j} - 3\sin t\vec{k}.$$
$$|\vec{r}\,'(t)| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5.$$
$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{3}{5}\cos t\vec{i} + \frac{4}{5}\vec{j} - \frac{3}{5}\sin t\vec{k}.$$

b)(5 pt) Find the unit normal vector.
solution)

$$\vec{T}\,'(t) = -\frac{3}{5}\sin t\vec{i} - \frac{3}{5}\cos t\vec{k}.$$
$$\left|\vec{T}\,'(t)\right| = \frac{3}{5}.$$
$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\left|\vec{T}\,'(t)\right|} = -\sin t\vec{i} - \cos t\vec{k}.$$

**Problem 6**(9 pt) A particle moves with constant speed, that is, $|\vec{v}(t)| = c$. Show that the velocity vector is orthogonal to the acceleration vector. **And** find the tangential component of acceleration.
solution)

$$|\vec{v}(t)| = c.$$
$$|\vec{v}(t)|^2 = c^2.$$
$$\vec{v}(t)\cdot\vec{v}(t) = c^2.$$
$$\vec{v}\,'(t)\cdot\vec{v}(t) + \vec{v}(t)\cdot\vec{v}\,' = 0.$$
$$2\vec{v}\,'(t)\cdot\vec{v}(t) = 0.$$
$$\vec{a}(t)\cdot\vec{v}(t) = 0.$$

Hence, $\vec{v}(t) \perp \vec{a}(t)$ for all $t$.

Since the acceleration vector is perpendicular to the velocity vector, the tangential component of acceleration is 0.

$$\left(a_T = \vec{T}(t)\cdot\vec{a}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\cdot\vec{a}(t) = 0, \text{since } \vec{r}\,'(t)\cdot\vec{a}(t) = \vec{v}(t)\cdot\vec{a}(t) = 0.\right)$$