MATH 242C EXAM 2

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Problem 1 Let $f(x,y,z) = \ln(x+y-z^2)$. Use the fact that $f_x = \dfrac{1}{x+y-z^2}$, $f_y = \dfrac{1}{x+y-z^2}$, $f_z = \dfrac{-2z}{x+y-z^2}$, do the following problems.

a)(3 pt) Find the gradient vector of $f(x,y,z)$.

solution)

$$\vec{\nabla} f = \left\langle \frac{1}{x+y+z^2}, \frac{1}{x+y+z^2}, \frac{2z}{x+y+z^2} \right\rangle.$$

b)(3 pt) Find a vector in the direction in which f increase most rapidly at $(1,1,1)$.

solution)

$$\vec{\nabla} f(1,1,1) =< 1,1,-2 > .$$

c)(3 pt) Find the maximum rate of increase of f at $(1,1,1)$.

solution)

$$\left|\vec{\nabla} f(1,1,1)\right| = \sqrt{1+1+4} = \sqrt{6}.$$

d)(5 pt) Find the directional derivative of f at $(1,1,1)$ toward the point $(3,2,1)$.

solution)The vector in the direction from $(1,1,1)$ to $(3,2,1)$ is $< 3-1, 2-1, 1-1 >=< 2,1,0 >$. So, the unit vector in the same direction is

$$\vec{u} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle.$$

$$D_{\vec{u}} f = \vec{\nabla} f(1,1,1) \cdot \vec{u} =< 1,1,-1 > \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle = \frac{3}{\sqrt{5}}.$$

e)(5 pt) Find the linear approximation of f at $(1,1,1)$.

solution)

$$L(x,y,z) = f(1,1,1) + f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1)$$
$$= \ln(1) + (x-1) + (y-1) - 2(z-1) = x + y - 2z.$$

Problem 2(6 pt) Use implicit differentiation theorem to find $\dfrac{\partial z}{\partial y}$ at $(3,2,5)$ if $\dfrac{x^2}{9}+\dfrac{y^2}{4}-\dfrac{z^2}{25}=1$.

solution)

$$F(x,y,z)=\frac{x^2}{9}+\frac{y^2}{4}-\frac{z^2}{25}.$$

We know that

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.$$

So,

$$\frac{\partial z}{\partial y}=-\frac{\frac{y}{2}}{\frac{-2z}{25}}=\frac{25}{4}\frac{y}{z}.$$

$$\left.\frac{\partial z}{\partial y}\right|_{(3,2,5)}=\frac{25}{4}\frac{2}{5}=\frac{5}{2}.$$

Problem 3(7 pt) Verify that $z=f(\frac{y}{x})$ satisfies $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$.

solution) Let $u=\dfrac{y}{x}$. Using the chain rule, we get

$$\frac{\partial z}{\partial x}=\frac{dz}{du}\frac{\partial u}{\partial x}, \tag{1}$$

$$\frac{\partial z}{\partial y}=\frac{dz}{du}\frac{\partial u}{\partial y}. \tag{2}$$

And

$$\frac{\partial u}{\partial x}=-\frac{y}{x^2}, \tag{3}$$

$$\frac{\partial u}{\partial y}=\frac{1}{x}. \tag{4}$$

Plug in (3) into (1), we get

$$\frac{\partial z}{\partial x}=-\frac{y}{x^2}\frac{dz}{du}.$$

Plug (4) into (2), we have

$$\frac{\partial z}{\partial y}=\frac{1}{x}\frac{dz}{du}.$$

Hence,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{y}{x}\frac{dz}{du}+\frac{y}{x}\frac{dz}{du}=0.$$

Problem 4(9 pt) Find all the critical points of $f(x, y) = x^3 - 9xy + y^3$ **and** classify the critical points.

solution)

$$\begin{cases} f_x = 3x^2 - 9y = 0 \\ f_y = -9x + 3y^2 = 0. \end{cases}$$

$$\Longrightarrow \begin{cases} y = \dfrac{1}{3}x^2 \\ -9x + 3y^2 = 0. \end{cases}$$

$$\Longrightarrow -9x + 3\left(\frac{1}{3}x^2\right)^2 = -9x^2 + \frac{1}{3}x^4 = 0.$$

$$\Longrightarrow x(-27 + x^3) = 0.$$

$$x = 0, \quad 3.$$

Corresponding y-values are $y = 0$, and $y = 3$. Hence, the critical points are $(0, 0)$, and $(3, 3)$ Now,

$$D = f_{xx}f_{xx} - f_{xy}^2 = (6x)(6y) - (-9)^2 = 36xy - 81.$$

(x, y)	(0,0)	(3,3)
$D = 36xy - 81$	-81(-)	+
f_{xx}	0	+
	saddle	local minimum

Problem 5(9 pt) Use the Lagrange multiplier to find the point on $x+y+z=1$ that is closest to the origin $(0,0,0)$.

solution) Rewrite the problem as follows:
Minimize $f(x,y,z)=x^2+y^2+z^2$, subject to the constraint $g(x,y,z)=x+y+z=1$.

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = x+y+z=1. \end{cases}$$

$$\Longrightarrow \begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \\ x+y+z=1. \end{cases}$$

$$\Longrightarrow \begin{cases} \lambda = 2x = 2y = 2z \\ x+y+z=1. \end{cases}$$

$$\Longrightarrow \begin{cases} x = y = z \\ x+y+z=1. \end{cases}$$

$$\Longrightarrow 3x = 1, \quad x = \frac{1}{3} = y = z.$$

Hence, The point on the plane $x+y+z=1$ that is closest to $(0,0,0)$ is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.