

MATH 242C EXAM 2

NAME: Pink, Purple

Problem 1 Let $f(x, y, z) = \ln(x^2 - y + z)$. Use the fact that $f_x = \frac{2x}{x^2 - y + z}$, $f_y = \frac{-1}{x^2 - y + z}$, $f_z = \frac{1}{x^2 - y + z}$, do the following problems.

a)(3 pt) Find the gradient vector of $f(x, y, z)$.
solution)

$$\vec{\nabla} f = \left\langle \frac{2x}{x^2 - y + z}, \frac{-1}{x^2 - y + z}, \frac{1}{x^2 - y + z} \right\rangle.$$

b)(3 pt) Find a vector in the direction in which f increase most rapidly at $(2, 1, -2)$.
solution)

$$\vec{\nabla} f(2, 1, -2) = \langle 4, -1, 1 \rangle.$$

c)(3 pt) Find the maximum rate of increase of f at $(2, 1, -2)$.
solution)

$$\left| \vec{\nabla} f(2, 1, -2) \right| = \sqrt{16 + 1 + 1} = \sqrt{18}.$$

d)(5 pt) Find the directional derivative of f at $(2, 1, -2)$ toward the point $(3, 2, 5)$.

solution) The vector in the direction from $(2, 1, -2)$ to $(3, 2, 5)$ is $\langle 3 - 2, 2 - 1, 5 - (-2) \rangle = \langle 1, 1, 7 \rangle$. So, the unit vector in the same direction is

$$\vec{u} = \left\langle \frac{1}{\sqrt{51}}, \frac{1}{\sqrt{51}}, \frac{7}{\sqrt{51}} \right\rangle.$$

$$D_{\vec{u}} f = \vec{\nabla} f(2, 1, -2) \cdot \vec{u} = \langle 4, -1, 1 \rangle \cdot \left\langle \frac{2}{\sqrt{51}}, \frac{1}{\sqrt{51}}, \frac{7}{\sqrt{51}} \right\rangle = \frac{10}{\sqrt{51}}.$$

e)(5 pt) Find the linear approximation of f at $(2, 1, -2)$.
solution)

$$\begin{aligned} L(x, y, z) &= f(2, 1, -2) + f_x(2, 1, -2)(x-2) + f_y(2, 1, -2)(y-1) + f_z(2, 1, -2)(z+2) \\ &= \ln(1) + 4(x-2) - (y-1) + 2(z+2) = 4x - y + z - 5. \end{aligned}$$

Problem 2(6 pt) Use implicit differentiation theorem to find $\frac{\partial z}{\partial x}$ at (2, 3, 8) if $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{64} = 1$.
 solution)

$$F(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{64}.$$

We know that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.$$

So,

$$\frac{\partial z}{\partial x} = -\frac{\frac{x}{2}}{\frac{-z}{32}} = \frac{16x}{z}.$$

$$\left. \frac{\partial z}{\partial x} \right|_{(2,3,8)} = \frac{32}{8} = 4.$$

Problem 3(7 pt) Verify that $z = f\left(\frac{y}{x}\right)$ satisfies $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.
 solution) Let $u = \frac{y}{x}$. Using the chain rule, we get

$$(1) \quad \frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x},$$

$$(2) \quad \frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y}.$$

And

$$(3) \quad \frac{\partial u}{\partial x} = -\frac{y}{x^2},$$

$$(4) \quad \frac{\partial u}{\partial y} = \frac{1}{x}.$$

Plug in (3) into (1), we get

$$\frac{\partial z}{\partial x} = -\frac{y}{x^2} \frac{dz}{du}.$$

Plug (4) into (2), we have

$$\frac{\partial z}{\partial y} = \frac{1}{x} \frac{dz}{du}.$$

Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\frac{y}{x} \frac{dz}{du} + \frac{y}{x} \frac{dz}{du} = 0.$$

Problem 4(9 pt) Find all the critical points of $f(x, y) = x^2 + \frac{1}{3}y^3 - 2xy - 3y$ and classify the critical points.
 solution)

$$\begin{cases} f_x = 2x - 2y = 0 \\ f_y = y^2 - 2x - 3 = 0. \end{cases}$$

$$\implies \begin{cases} y = x \\ y^2 - 2x - 3 = 0. \end{cases}$$

$$\implies x^2 - 2x - 3 = (x - 3)(x + 1) = 0.$$

$$x = 3, \quad -1.$$

Corresponding y -values are $y = 3$, and $y = -1$. Hence, the critical points are $(3, 3)$, and $(-1, -1)$ Now,

$$D = f_{xx}f_{xx} - f_{xy}^2 = 4y - (-2)^2 = 4y - 4.$$

(x, y)	$(3,3)$	$(-1,-1)$
$D = 36xy - 81$	$8(+)$	$-8(-)$
f_{xx}	$+$	$+$
	local minimum	saddle

Problem 5(9 pt) Use the Lagrange multiplier to find the point on $x + 2y + z = 1$ that is closest to the origin $(0, 0, 0)$. solution) Rewrite the problem as follows:

Minimize $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraint $g(x, y, z) = x + y + z = 1$.

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x, y, z) = x + 2y + z = 1. \end{cases}$$

$$\implies \begin{cases} 2x = \lambda \\ 2y = 2\lambda \\ 2z = \lambda \\ x + 2y + z = 1. \end{cases}$$

$$\implies \begin{cases} \lambda = 2x = y = 2z \\ x + y + z = 1. \end{cases}$$

$$\implies \begin{cases} y = 2x, z = x \\ x + 2y + z = 1. \end{cases}$$

$$\implies 6x = 1, \quad x = \frac{1}{6} = z, \quad y = \frac{1}{3}.$$

Hence, The point on the plane $x + 2y + z = 1$ that is closest to $(0, 0, 0)$ is $(\frac{1}{6}, \frac{1}{3}, \frac{1}{6})$.