# Math 242 C EXAM 3

**NAME:** **Type B**

**Problem 1** Evaluate the integrals.

a) $\int_0^1 \int_1^4 (3 - x)dydx$

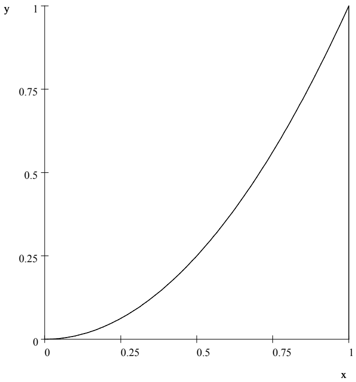
$$\begin{aligned}\int_0^1 \int_1^4 (3 - x)dydx &= \int_0^1 [3y - xy]_{y=1}^{y=4} dx \\ &= \int_0^1 (9 - 3x)dx = \frac{15}{2}.\end{aligned}$$

b) $\int_0^{\frac{\pi}{4}} \int_0^{2\cos\theta} rdrd\theta$ (You may use the formula $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ or $\sin^2\theta = \frac{1-\cos 2\theta}{2}$)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \int_0^{2\cos\theta} rdrd\theta &= \int_0^{\frac{\pi}{4}} \left[\frac{1}{2}r^2\right]_{r=0}^{r=2\cos\theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} 2\cos^2\theta d\theta = \int_0^{\frac{\pi}{4}} (1 + \cos 2\theta)d\theta \\ &= \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=\frac{\pi}{4}} = \frac{\pi}{4} + \frac{1}{2}.\end{aligned}$$

c) $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dxdy$ (Hint: Change the order of integration.)

Note that the region of integration is $\left\{(x,y) : \sqrt{y} \leq x \leq 1,\ 0 \leq y \leq 1\right\}$. Sketching the region, we have

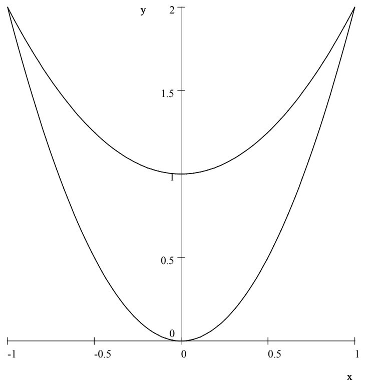


Reading off the region of integration again, we see that $0 \leq y \leq x^2,\ \ 0 \leq x \leq 1$. Hence,

$$\begin{aligned}
\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dxdy &= \int_0^1 \int_0^{x^2} e^{x^3} dydx \\
&= \int_0^1 \left[ye^{x^3}\right]_{y=0}^{y=x^2} dx = \int_0^1 x^2 e^{x^3} dx \\
&= \int_{x=0}^{x=1} \frac{1}{3} e^u du, \qquad u = e^{x^3},\ \ du = 3x^2 dx \\
&= \left[\frac{1}{3}e^u\right]_{x=0}^{x=1} = \left[\frac{1}{3}e^{x^3}\right]_{x=0}^{x=1} = \frac{1}{3}(e-1).
\end{aligned}$$

**Note: In what follows, I will denote regions in plane by $D$ and solids in space by $E$.**

**Problem 2** Express the area of the region bounded by $y = 1 + x^2$ and $y = 2x^2$ as an iterated **double integral**. *Do not evaluate.*

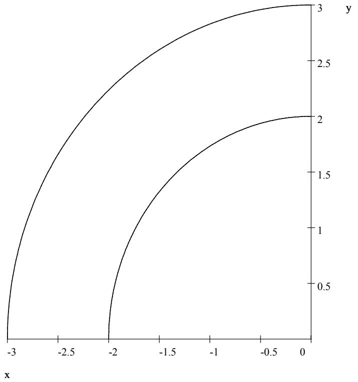


Intersections: $1 + x^2 = 2x^2 \Rightarrow x = -1, 1$. So we can express the region as $\left\{(x,y) : \ -1 \le x \le 1, \ x^2 \le y \le 1 + x^2\right\}$.

Area of the region $D$ is $\iint\limits_D dA$. So,

$$A(D) = \int_{-1}^{1} \int_{x^2}^{1+x^2} dydx.$$

**Problem 3** Express the area of the region in the second quadrant bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$ as an iterated **double integral in polar coordinates**. *Do not evaluate.*

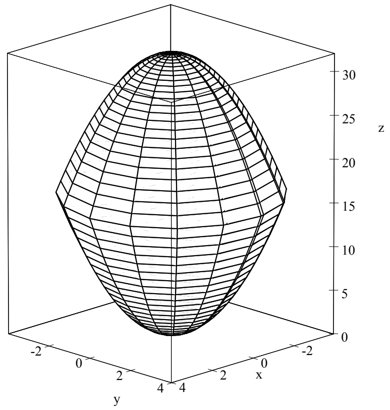


The region can be expressed as $\left\{(r,\theta) : \ 2 \le r \le 3, \ \frac{\pi}{2} \le \theta \le \pi\right\}$. Note that the area element $dA$ becomes $rdrd\theta$ in polar coordinates.

So,

$$A(D) = \int_{\frac{\pi}{2}}^{\pi} \int_{2}^{3} rdrd\theta.$$

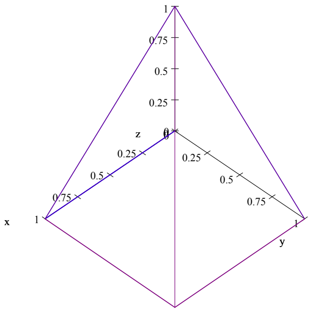
**Problem 4** Express the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 32 - x^2 - y^2$ as an iterated **triple integral**. *Do not evaluate.*

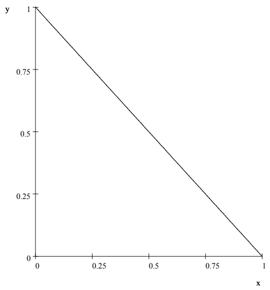


The projection of the solid onto $xy$ –plane is $\{(x,y) : x^2 + y^2 \le 16\}$, since $x^2 + y^2 = 32 - x^2 - y^2 \Rightarrow x^2 + y^2 = 16$.
So, the region $E$ is
$\{(x,y,z) : -4 \le x \le 4, -\sqrt{16 - x^2} \le y \le \sqrt{16 - x^2}, x^2 + y^2 \le z \le 32 - x^2 - y^2\}$
Also note that the volume of the solid $E$ is $V(E) = \int\int\int_E dV$. Hence,

$$V(E) = \iiint\limits_E dV = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{x^2+y^2}^{32-x^2-y^2} dzdydx.$$

**Problem 5** Express $\int\int\int_E xdV$ as an iterated integrals, where $E$ is the region bounded by the planes $x = 0,\ y = 0,\ z = 0,\ y + z = 1,$ and $x + z = 1.$ *Do not evaluate.*





**Projection onto** $yz$ **–plane**: $\{(y,z)\ :\ 0 \le y \le 1 - z, 0 \le z \le 1\}$

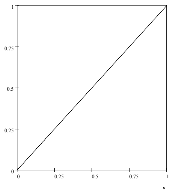
We will project the solid onto $yz$ –plane. Then we get the region bounded by the lines $x = 0,$ $y = 0,$ and $y + z = 1.$ Since we project the region onto $yz$ –plane, we need to determine the range for $x$.
We see from the graph (or the equtions given) that $0 \le x \le 1 - z.$
Hence, the solid is

$$\{(x,y,z)\ :\ 0 \le x \le 1 - z, 0 \le y \le 1 - z, 0 \le z \le 1\}.$$

Thus,

$$\iiint_E xdV = \int_0^1 \int_0^{1-z} \int_0^{1-z} xdxdydz.$$

**Remark:** Several different answers are possible in this problem: if you project onto $xz$ –plane, one possible solution is $\int_0^1 \int_0^{1-z} \int_0^{1-z} xdydxdz.$
If you project onto $xy$ –plane, it gets complicated. We need to divide the square on $xy$-plane into upper half triangle and lower half triangle.

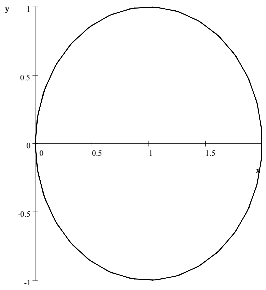


The lower half triangle is the projection of the plane $x + z = 1$ and upper half triangle is the projection of the plane $y + z = 1.$ So,

$$\iiint_E dV = \int_0^1 \int_y^1 \int_0^{1-x} xdzdxdy + \int_0^1 \int_0^y \int_0^{1-y} xdzdxdy.$$

Also, we should be careful choosing the right projection: for example, if we mistakenly use the lower triangle for the projection of the plane $y + z = 1,$ we get $\int_0^1 \int_y^1 \int_0^{1-y} xdzdxdy.$
But $\int_0^1 \int_0^y \int_0^{1-y} xdzdxdy = \frac{1}{24}$, while $\int_0^1 \int_y^1 \int_0^{1-y} xdzdxdy = \frac{5}{24}$.

**Problem 6** Express the surface area of the part of the paraboloid $z = \sqrt{4 - x^2 - y^2}$ that lies within the cylinder $x^2 + y^2 = 2x$ as an iterated double integral **in polar coordinates**. *Do not evaluate.*



Recall the surface area formula: $A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$.

Here, .

$$f(x,y) = \sqrt{4 - x^2 - y^2} \Rightarrow f_x = \frac{-x}{\sqrt{4 - x^2 - y^2}},\ f_y = \frac{-y}{\sqrt{4 - x^2 - y^2}} \Rightarrow f_x = -\frac{r\cos\theta}{\sqrt{4 - r^2}}, f_y = -\frac{r\sin\theta}{\sqrt{4 - r^2}}.$$

Note that $x^2 + y^2 = 2x \Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$.

So, in polar coordinates, we write the domain $D$ as

$$D = \left\{(r,\theta) : 0 \le r \le 2\cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}.$$

Hence, the surface area is

$$\begin{aligned}
A(S) &= \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \sqrt{1 + \frac{r^2\cos^2\theta}{4 - r^2} + \frac{r^2\sin^2\theta}{4 - r^2}}\ r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \sqrt{\frac{4 - r^2 + r^2\cos^2\theta + r^2\sin^2\theta}{4 - r^2}}\ r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \sqrt{\frac{4}{4 - r^2}}\ r dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{2}{\sqrt{4 - r^2}}\ r dr d\theta.
\end{aligned}$$