Practice Problems for the Final

Problem 1 Let $\vec{a} = < 1,0,1 >$, $\vec{b} = < 0,1,1 >$ and $\vec{c} = < 1,1,1 >$.

a) Compute the angle between $\vec{a}$ and $\vec{b}$.

solution)

$$\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta \Rightarrow 1 = (\sqrt{2})(\sqrt{2})\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

b) Find a vector which is perpendicular to both $\vec{a}$ and $\vec{b}$.

solution)

We know that either $\vec{a} \times \vec{b}$ or $\vec{b} \times \vec{a}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\vec{i} - \vec{j} + \vec{k}.$$

c) Find(ifpossible) a vector $\vec{d}$ so that $\vec{b} \cdot \vec{c} = \vec{d}$. If there is none , explain your answer.

solution)

There is no such a vector $\vec{d}$, since the dot procudt results in a scalar (number).

Problem 2 Find the parametric equation of the line which is perpendicular to the plane $x + 2y + 3z = 4$ and passes through $(-1,-2,-3)$.

solution)

The normal vector to the plane is $< 1,2,3 >$. So a directional vector for the line is $< 1,2,3 >$. Hence, the parametric equation for the line is

$$x = -1 + t,\ t \in R$$

$$y = -2 + 2t,\ t \in R$$

$$z = -3 + 3t,\ t \in R.$$

Problem 3 Let $F(x,y,z) = xe^{yz}$.
a) Find the gradient of F at $(-2,1,1)$.

solution)

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$$

$$\nabla F(-2,1,1) = \langle e, -2e, -2e \rangle.$$

b) Find the directional derivative of F in the direction of $< 1.-2.3 >$ at $(-2,1,1)$.

solution) The unit vector along $< 1.-2.3 >$ is $\vec{u} = \frac{1}{\sqrt{14}}(1,-2,3 >$. Hence the directional derivative is

$$D_{\vec{u}}F(-2,1,1) = \nabla F(-2,1,1) \cdot \frac{1}{\sqrt{14}}(1,-2,3 >$$

$$= < e,-2e,-2e > \cdot \frac{1}{\sqrt{14}}(1,-2,3 >$$

$$= \frac{1}{\sqrt{14}}(e + 4e - 6e) = -\frac{e}{\sqrt{14}}.$$

c) Find the equation of the tangent plane to the surface $xe^{yz} = -2e$ at $(-2,1,1)$.

solution) We know that the equation of the tangent plane to the surface $F(x,y,z) = c$ (c : constant) at (x_0,y_0,z_0) is

$$F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0.$$

So,

$$e(x+2) - 2e(y-1) - 2e(z-1) = 0 \Rightarrow (x+2) - 2(y-1) - 2(z-1) = 0 \Rightarrow x - 2y - 2z = -6.$$

d) Find the rate of change of z with respect to y if $xe^{yz} = -2e$.

solution) First, we nee to know that the rate of change of z with respect to y means $\frac{\partial z}{\partial y}$.
Let $F(x,y,z) = xe^{yz} + 2e = 0$.
Using implicit differentiation theorem, we get

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xze^{yz}}{xye^{yz}} = -\frac{z}{y}.$$

Problem 4 Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) = (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

a) Compute $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$ by using polar coordinates.

solution)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r} = \lim_{r\to 0} r\cos\theta\sin\theta = 0.$$

b) Is $f(x,y)$ continuous at $(0,0)$? Explain your answer.
solution) $f(x,y)$ is continuous at $(0,0)$, since $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

Problem 5 Find the absolute maximum and minimum values of $f(x,y) = 2x^2 + x + y^2 - 2$ on the region $D = \{(x,y) | \, x^2 + y^2 \leq 4\}$.

solution) Note that the max/min occurs either at the critical points in $\{(x,y) | \, x^2 + y^2 < 4\}$ or on the boundary $x^2 + y^2 = 4$.

1) Critical points:

$$f_x = 4x + 1 = 0 \Rightarrow x = -\frac{1}{4}$$
$$f_y = 2y = 0 \Rightarrow y = 0.$$

2) To find the max/min on $x^2 + y^2 = 4$, let's use the Lagrange multipliers method.

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x,y) &= c \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} 4x + 1 &= 2\lambda x \\ 2y &= 2\lambda y \\ x^2 + y^2 &= 4 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} 4x + 1 &= 2\lambda x \\ y &= 0 \text{ or } \lambda = 1 \\ x^2 + y^2 &= 4 \end{aligned}$$

$\Rightarrow$

$$\begin{cases} y = 0 \\ x^2 + y^2 = 4 \end{cases} \quad \text{Or,} \quad \begin{cases} 4x + 1 = 2x \\ x^2 + y^2 = 4 \end{cases}$$

$$\Rightarrow \{x = 2, -2, y = 0 \quad \text{Or,} \quad \begin{cases} x = -\frac{1}{2} \\ \frac{1}{4} + y^2 = 4 \Rightarrow y = \pm\frac{\sqrt{15}}{2} \end{cases}$$

Now, collect all the points (x,y) in step 1), 2).

(x,y)	$(-\frac{1}{4},0)$	$(2,0)$	$(-2,0)$	$(-\frac{1}{2},\frac{\sqrt{15}}{2})$	$(-\frac{1}{2},-\frac{\sqrt{15}}{2})$
$f(x,y)$	$-\frac{17}{8}$	8	4	$\frac{7}{4}$	$\frac{7}{4}$
	minimum	maximum			

Problem 6

a) Evaluate the integral $\int_0^1\int_0^x y\sqrt{x^2+y^2}\,dydx$.

solution)

$$\int_0^1\int_0^x y\sqrt{x^2+y^2}\,dydx = \int_0^1\int_{y=0}^{y=x}\frac{1}{2}\sqrt{u}\,dudx, \quad u = x^2+y^2, du = 2ydy \text{ (keeping } x \text{ fixed)}$$

$$= \int_0^1\left[\frac{1}{3}u^{\frac{3}{2}}\right]_{y=0}^{y=x}dx = \int_0^1\left[\frac{1}{3}(x^2+y^2)^{\frac{3}{2}}\right]_{y=0}^{y=x}dx$$

$$= \int_0^1\left(\frac{1}{3}(2x^2)^{\frac{3}{2}} - \frac{3}{4}x^3\right)dx = \frac{1}{3}(2^{\frac{3}{2}}-1)\int_0^1 x^3dx$$

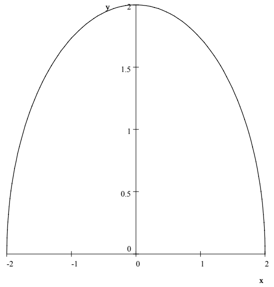
$$= \frac{\left(2\sqrt{2}-1\right)}{3}\left[\frac{1}{4}x^4\right]_0^1 = \frac{2\sqrt{2}-1}{12} = \frac{1}{6}\sqrt{2} - \frac{1}{12}.$$

b) Evaluate $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2dxdy$ by converting to polar coordinates.

solution)

$$-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$$
$$0 \le y \le 2$$

This is the upper half disk with radius 2 and center $(0,0)$.



So,

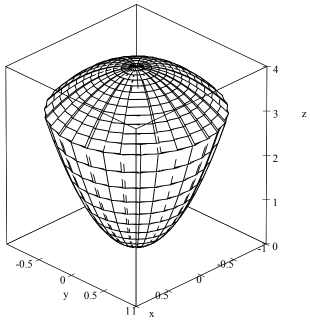
$$0 \le r \le 2,\ \ 0 \le \theta \le \pi,\ dxdy = rdrd\theta.$$

$$\begin{aligned}\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 dxdy &= \int_0^{\pi} \int_0^2 r^3 \cos^2\theta drd\theta \\ &= \int_0^2 r^3 dr \int_0^{\pi} \cos^2\theta d\theta \\ &= \left[\frac{1}{4} r^4\right]_0^2 \frac{1}{2} \int_0^{\pi} (1-\cos 2\theta) d\theta \\ &= 4 \times \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = 2\pi.\end{aligned}$$

Problem 7

a) Express the volume of the solid above the surface $z = 3x^2 + 3y^2$ and below the surface $z = 4 - x^2 - y^2$ in cylindrical coordinates. Do not evaluate.

solution)



Intersection of two surfaces is

$$3x^2 + 3y^2 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1.$$

So the projection onto xy-plane is the disk: $x^2 + y^2 \le 1$.
Also note that upper one is $z = 4 - x^2 - y^2$ and the lower one is $z = 3x^2 + 3y^2$.
Hence,

$$3x^2 + 3y^2 \le z \le 4 - x^2 - y^2$$

$$(x,y) \in D : x^2 + y^2 \le 1 \Rightarrow 0 \le r \le 1, 0 \le \theta \le 2\pi.$$

In cylindrical coordinates, we have

$$3r^2 \le z \le 4 - r^2$$

$$0 \le r \le 1$$

$$0 \le \theta \le 2\pi.$$

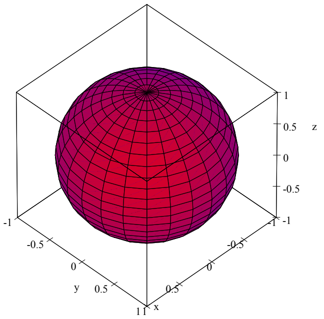
And

$$dV = rdzdrd\theta.$$

Thus,

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_{3r^2}^{4-r^2} r dz dr d\theta$$

b) Express the volume of the sphere $\rho = 1$ in spherical coordinates. Do not evaluate.
solution)



We can express the sphere $\rho = 1$ as follows:

$$0 \le \rho \le 1$$
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi.$$

In spherical coordinates, we have

$$dV = \rho^2 \sin\phi d\rho d\phi d\theta.$$

Hence,

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi d\rho d\phi d\theta.$$

Problem 8

a) Convert $\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dzdydx$ to cylindrical coordinates.

solution)
We see from the limits of integration that

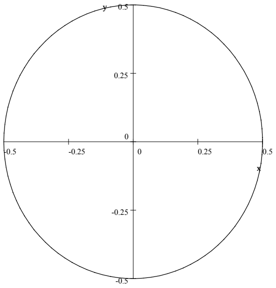
$$\sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}$$
$$-\sqrt{\frac{1}{2}-x^2} \le y \le \sqrt{\frac{1}{2}-x^2}$$
$$-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$

Substituting $x = r\cos\theta, y = r\sin\theta$, we get

$$r \le z \le \sqrt{1-r^2}.$$

To determine (r, θ), we read the inequalities for x and y and sketch the region: it is not hard to see that it is a disk with radius $\frac{1}{\sqrt{2}}$:

$y = \pm\sqrt{\frac{1}{2} - x^2} \Rightarrow x^2 + y^2 = \frac{1}{2}$



Hence,

$$r \le z \le \sqrt{1 - r^2}$$
$$0 \le r \le \frac{1}{\sqrt{2}}$$
$$0 \le \theta \le 2\pi.$$

So,

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dzdydx = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_r^{\sqrt{1-r^2}} rdzdrd\theta.$$

:

b) Convert $\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dzdydx$ to spherical coordinates.

solution)

Note that direct substitution will not give us an satisfactory answer in this problem.

Again, we have

$$\sqrt{x^2 + y^2} \le z \le \sqrt{1 - x^2 - y^2}$$
$$-\sqrt{\frac{1}{2} - x^2} \le y \le \sqrt{\frac{1}{2} - x^2}$$
$$-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$

Imagine Ice cream cone.

Note that

$$z = \sqrt{x^2 + y^2} \Rightarrow \rho\cos\theta = \rho\sin\phi \Rightarrow \tan\theta = 1 \Rightarrow \phi = \frac{\pi}{4}$$
$$z = \sqrt{1 - x^2 - y^2} \Rightarrow z^2 = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1.$$

Note also that inequalities: $-\sqrt{\frac{1}{2} - x^2} \le y \le \sqrt{\frac{1}{2} - x^2}, -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$ represent the disk on the right half plane.

So,

$$0 \le \theta \le 2\pi.$$

Hence, we have

$$0 \leq \rho \leq 1$$
$$0 \leq \phi \leq \frac{\pi}{4}$$
$$0 \leq \theta \leq 2\pi.$$

Thus,

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dzdydx = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin\phi d\rho d\phi d\theta$$

Problem 9 Evaluate $\int\int_D xydxdy,$ where D is the region in the first quadrant bounded by the curves $x^2 + y^2 = 4,\ x^2 + y^2 = 9, x^2 - y^2 = 1, x^2 - y^2 = 4.$
solution)
The boundaries of the region suggest that we set

$$u = x^2 + y^2,\ \ v = x^2 - y^2.$$

So,

$$4 \leq u \leq 9,\ \ 1 \leq v \leq 4.$$

We want x and y in terms of u and v.
Since

$$u + v = 2x^2,\ \ u - v = 2y^2 \Rightarrow x = \sqrt{\frac{u+v}{2}},\ \ y = \sqrt{\frac{u-v}{2}}.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{2}}{4\sqrt{u+v}} & \frac{\sqrt{2}}{4\sqrt{u+v}} \\ \frac{\sqrt{2}}{4\sqrt{u-v}} & -\frac{\sqrt{2}}{4\sqrt{u-v}} \end{vmatrix} = -\frac{1}{4\sqrt{u^2 - v^2}}.$$

Hence,

$$\int\int_D xydxdy = \int_1^4 \int_4^9 \sqrt{\frac{u+v}{2}} \sqrt{\frac{u-v}{2}} \frac{1}{4\sqrt{u^2 - v^2}} dudv = \int_1^4 \int_4^9 \frac{1}{8} dudv = \frac{15}{8}.$$