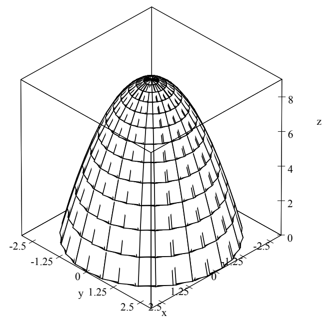
Practice Problems for Section 15.8 and 15.9

Problem 1 Express $\iiint_E \sqrt{x^2+y^2}$ in cylindrical coordinates, where E is the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

solution)



The paraboloid intersects the xy-plane in the circle $x^2+y^2 = 9$. Hence the projection D onto xy-plane is .

$$D = \{(x,y) : x^2+y^2 \le 9\}.$$

In polar coordinates, we have

$$D = \{(r,\theta) : 0 \le r \le 3, 0 \le \theta \le 2\pi\}.$$

Hence,

$$E = \{(x,y,z) : (x,y) \in D, 0 \le z \le 9 - x^2 - y^2\}.$$

In cylindrical coordinates,

$$E = \{(r,\theta,z) : 0 \le r \le 3, 0 \le \theta \le 2\pi, 0 \le 9 - r^2\}.$$

Now, we know $\sqrt{x^2+y^2} = r$ and $dA = rdzdrd\theta$ in cylindrical coordinates.

Thus,

$$\iiint_E \sqrt{x^2+y^2} = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2 dzdrd\theta.$$

Problem 2 Convert $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} (xyz) dzdxdy$ to cylindrical coordinates.

solution)

Reading off the limits of integration, we know

$$x^2 + y^2 \le z \le \sqrt{x^2+y^2} \Rightarrow r^2 \le z \le r.$$

$$0 \le x \le \sqrt{1-y^2}, \quad 0 \le y \le 1 \Rightarrow 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 1.$$

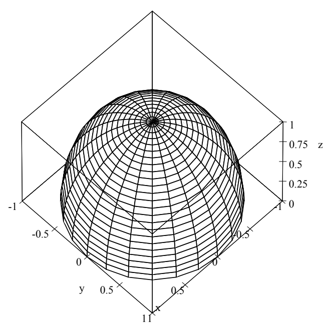
Substituing $x = r\cos\theta, y = r\sin\theta$, and $z = z$, the integrand xyz becomes $(r\cos\theta)(r\sin\theta)z = zr^2 \sin\theta\cos\theta$. And $dzdxdy = rdzdrd\theta$ in cylindrical coordinates.

Thus,

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} (xyz) dzdxdy = \int_0^{\frac{\pi}{2}} \int_0^1 \int_{r^2}^{r} r^3 \cos\theta \sin\theta \, zdzdrd\theta.$$

Problem 3 Express $\iiint_E xyzdV$ in spherical coordinates , where E is $0 \le z \le \sqrt{1-x^2-y^2}$.

solution) Note that $z = \sqrt{1-x^2-y^2}$ $(\Rightarrow z^2 = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 1)$ is the upper hemi -sphere.



So,

$$E: \quad 0 \le \rho \le 1, \ 0 \le \phi \le \frac{\pi}{2}, \ 0 \le \theta \le 2\pi.$$

Substituting $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, the integrand becomes $xyz = \rho^3 \sin^2\phi\cos\phi\cos\theta\sin\theta$.

And $dV = \rho^2 \sin\phi d\rho d\phi d\theta$ in spherical coordinates.

Hence,

$$\iiint_E xyzdV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^5 \sin^3\phi\cos\phi\cos\theta\sin\theta d\rho d\phi d\theta.$$

Problem 4 Express the volume of the solid that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 4\cos\phi$.
solution)
Note that

$$\rho = 4\cos\phi \Rightarrow \rho^2 = 4\rho\cos\phi \Rightarrow x^2 + y^2 + z^2 = 4z \Rightarrow x^2 + y^2 + (z-2)^2 = 4.$$

The graph is similar to the figure 10 on page 1022 in the textbook.

$$0 \le \rho \le 4\cos\phi,\ 0 \le \phi \le \frac{\pi}{3},\ 0 \le \theta \le 2\pi.$$

Also note that $dV = \rho^2 d\rho d\phi d\theta$ in spherical coordinates.

$$V(E) = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{4\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta.$$

Problem 5 Let Q be the elliptical region $\frac{x^2}{4} + \frac{y^2}{9} \le 1$. Evaluate $\iint_Q x^2 dA$ by making an appropriate change of variables.
solution) Let $u = \frac{x}{2}(\Rightarrow x = 2u)$ and $v = \frac{y}{3}(\Rightarrow y = 3v)$. Then,

$$u^2 + v^2 \le 1.$$

Next,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6.$$

Hence

$$\iint_Q x^2 dA = \iint_{\{(u,v):u^2+v^2\le 1\}} (2u)^2(6)dudv = 24\int_0^{2\pi}\int_0^1 r^2\cos^2\theta r dr d\theta$$

$$= 24\int_0^{2\pi}\cos^2\theta d\theta \int_0^1 r^3 dr = 12\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{\theta=2\pi}\left[\frac{1}{4}r^4\right]_{r=0}^{r=1}$$

$$= 12 \cdot 2\pi \cdot \frac{1}{4} = 6\pi.$$

Problem 6 Evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right)dA,$ where R is the trapezoidal region with vertices $(1,0),(2,0),(0,2),(0,1)$.

solution) First, we need to decide the transformation.

Since it is not easy to integrate $\cos\left(\frac{y-x}{y+x}\right)$, use the following the change of variables:

$$u = y - x,\ \ v = y + x.$$

Then,

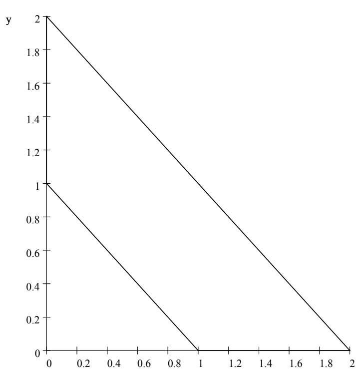
$$x = \frac{1}{2}(-u+v),\quad y = \frac{1}{2}(u+v)$$

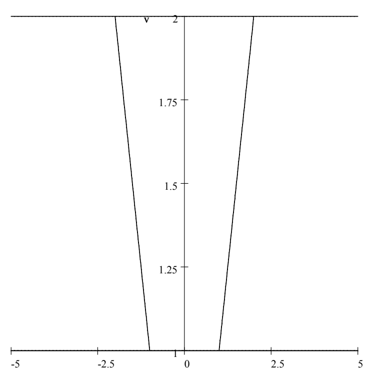
So,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2} \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

Note that R is the region bounded by the lines $x+y=1, x+y=2, x=0$ and $y=0$.
So,

$$x+y=1 \to v=1$$
$$x+y=2 \to v=2$$
$$x=0 \to u=y, v=y \Rightarrow u=v$$
$$y=0 \to u=-x, v=x \Rightarrow u=-v$$





$R_{xy} = R$ $\qquad\qquad\qquad\qquad$ R_{uv}

Now, we can easily see that the transformed region R_{uv} is

$$-v \le u \le v,\quad 1 \le v \le 2.$$

Hence,

$$\iint_{R_{xy}} \cos\left(\frac{y-x}{y+x}\right) dA = \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right) \frac{1}{2} . du dv$$

$$= \frac{1}{2} . \int_1^2 \left[v \sin\left(\frac{u}{v}\right) \right]_{u=-v}^{u=v} dv$$

$$= \frac{1}{2} \int_1^2 v \sin(1) - v \sin(-1) dv$$

$$= \frac{1}{2} (\sin(1) - \sin(-1)) \int_1^2 v dv$$

$$= \frac{1}{2} (\sin(1) - \sin(-1)) \frac{3}{2} = \frac{3}{4} \cdot 2 \sin 1 = \frac{3}{2} \sin 1, \quad \text{since } \sin(-1) = -\sin 1.$$