Problem 1 Let $\vec{a} =< -2, 0, 1 >$ and $\vec{b} =< 1, -3, 1 >$.
a) Compute $\vec{a} \cdot \vec{b}$.
solution) $\vec{a} \cdot \vec{b} = -2 + 0 + 1 = -1$.

b) Find the cosine of the angle between $\vec{a}$ and $\vec{b}$.
solution)

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{-1}{\sqrt{4+1}\sqrt{1+9+1}} = -\frac{1}{\sqrt{55}}.$$

c) Find the projection vector $\vec{a}$ onto $\vec{b}$.
solution)

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b} = -\frac{1}{11} < 1, -3, 1 >.$$

Problem 2 Compute $\vec{a} \times \vec{b}$ with $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = \vec{i} - 2\vec{j} - \vec{k}$.
solution)

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & -2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix}\vec{k} \\ &= 3\vec{i} + 2\vec{j} - \vec{k}. \end{aligned}$$

Problem 3 Let $\vec{u}$ and $\vec{v}$ be two nonzero vectors. Find $(\vec{u} \times \vec{v}) \cdot (\vec{u} + \vec{v})$.
solution)

$$\begin{aligned} &(\vec{u} \times \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &\quad = (\vec{u} \times \vec{v}) \cdot \vec{u} + (\vec{u} \times \vec{v}) \cdot \vec{v} \quad \text{by the distributive property of the dot product} \\ &\quad = 0 + 0 = 0, \quad \text{since } \vec{u} \times \vec{v} \text{ is orthogonal to both } \vec{u} \text{ and } \vec{v}. \end{aligned}$$

Problem 4 Find the parametric equation of the line which parallel to the line $\frac{x+1}{3} = y-1 = \frac{z+1}{2}$ and passes through $(2,3,1)$.

solution) Since the line is parallel to the line with directional vector $< 3,1,2 >$, we can choose a directional vector same as $< 3,1,2 >$. The parametric equation of the line with directional vector $< 3,1,2 >$ and a point $(2,3,1)$ is

$$x = 2+3t, \quad y = 3+t, \quad z = 1+t.$$

Problem 5 Find the vector equation of the line which passes through $(2,-4,6)$ and $(5,0,2)$.

solution) Since the points $P(2,-4,6)$ and $Q(5,0,2)$ are on the line, a directional vector is

$$\vec{PQ} =< 5-2, 0+4, 2-6 >=< 3,4,-4 >.$$

If we choose a point $(5,0,2)$, then the vector equation of the line is

$$< x-5, y-0, z-2 >= t < 3,4,-4 >, \quad t \in \mathbb{R}.$$

$$< x,y,z >=< 5,0,2 > +t < 3,4,-4 >, \quad t \in \mathbb{R}.$$

Problem 6 Find the equation of the plane that contains $(1,0,2)$, $(3,4,5)$, and $(0,0,1)$.

solution) Let $P(1,0,2)$, $Q(3,4,5)$, and $R(0,0,1)$. We need two nonparallel vectors on the plane to find the equation. Two non parallel vectors are

$$\vec{PQ} =< 3-1, 4-0, 5-2 >=< 2,4,3 >, \quad \vec{PR} =< 0-1, 0-0, 1-2 =< -1,0,-1 >.$$

Normal vector to the plane is

$$\vec{PQ} \times \vec{PR} =< 2,4,3 > \times < -1,0,-1 >= -4\vec{i} - \vec{j} + 4\vec{k}.$$

Use a point $R(0,0,1)$ and the normal vector $< -4,-1,4 >$, the equation of the plane is

$$-4(x-0) - (y-0) + 4(z-1) = 0.$$

Simplifying this, we get

$$4x + y - 4z = -4.$$

Problem 7 Find the equation of the plane containing the line $\frac{x-1}{2} = \frac{y-2}{-3} = z+1$ and passes through $(3,0,1)$.

solution) We can easily find two thing by reading off the equation of the line: a directional vector is $< 2,-3,1 >$. A point $(1,2,-1)$ is on the line.

Since the line is on the plane, $(1,2,-1)$ is also on the plane. What we need is two nonparallel vectors on the plane to find the equation of the plane: We can choose

$$\vec{a} =< 2,-3,1 >, \quad \text{and} \quad \vec{b} =< 1-3, 2-0, -1-1 >=< -2,2,-2 >.$$

So, the normal vector to the plane is

$$\vec{a} \times \vec{b} =< 2,-3,1 > \times < -2,2,-2 >= 4\vec{i} + 2\vec{j} - 2\vec{k}.$$

Hence, the equation of the plane is

$$4(x-3) + 2(y-0) - 2(z-1) = 0.$$

$$4x + 2y - 2z = 10.$$

$$2x + y - z = 5.$$

Problem 8 Show that $(-1,6,-5)$ is not on the plane $2x + y + z = 2$. And find the equation of the plane through $(-1,6,-5)$ and parallel to the plane $2x + y + z = 2$.

solution) Plug in $(-1,6,-5)$ into $2x + y + z = 2$. Then,

$$2(-1) + 6 - 5 = -1 \neq 2.$$

So, $(-1,6,-5)$ is not on the plane.

Normal vector to the plane is $< 2,1,1 >$. So the equation of the plane is

$$2(x+1) + (y-6) + (z+5) = 0.$$

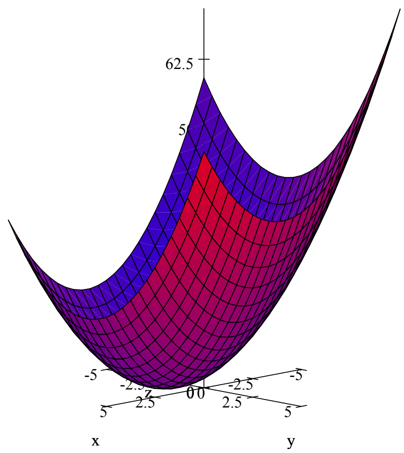
$$2x + y + z = -1.$$

Problem 9 Sketch the graph of $z = x^2 - 2x + y^2 + 2y + 2$

solution) By completing square for x and y, we get

$$z = (x-1)^2 + (y+1)^2.$$

It represents an paraboloid and has been shifted so that its vertex is $(1,-1,0)$.



Problem 10 Let $\vec{r}(t) =< t, 3, t^2 >$.
a) Find the unit tangent vector at $t = 1$.
solution)

$$\vec{r}'(t) =< 1, 0, 2t >, \quad \vec{r}'(1) =< 1, 0, 2 >.$$

$$|\vec{r}'(1)| = \sqrt{1+4} = \sqrt{5}.$$

The unit tangent vector $\vec{T}(t)$ at $t = 1$ is

$$\vec{T}(1) = \frac{\vec{r}'(1)}{|\vec{r}'(1)|} = \frac{1}{\sqrt{5}} < 1, 0, 2 >.$$

b) Find the parametric equation of the tangent line at $t = 1$.
solution) A directional vector is $\vec{r}'(1) =< 1, 0, 2 >$.
A point is $(1, 3, 1)$. So the equation of the tangent line at $t = 1$ is

$$x = 1 + t, \quad y = 3, \quad z = 1 + 2t.$$

Problem 11 A particle moves along $\vec{r}(t) =< 3\cos 2t, 3\sin 2t, 2t >$.
a) Find the speed $v(t)$. Is the speed constant?

solution)

$$\vec{v}(t) = r'(t) =< -6\sin 2t, 6\cos 2t, 2 >.$$

$$v(t) = |\vec{v}(t)| = \sqrt{36\sin^2 2t + 36\cos^2 2t + 4} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}.$$

So the speed is a constant.

b) Use part b) to show that $\vec{v}(t) \perp \vec{a}(t)$ for all t.

solution) From part a), $|\vec{v}(t)|^2 = 40$.
Note that $|\vec{v}(t)|^2 = \vec{v}(t) \cdot \vec{v}(t)$. So,

$$\vec{v}(t) \cdot \vec{v}(t) = 40.$$

Differentiating each side, we get

$$\vec{v}'(t) \cdot \vec{v}(t) + \vec{v}(t) \cdot \vec{v}'(t) = 0.$$

$$2\vec{v}(t) \cdot \vec{v}'(t) = 0.$$

Hence,

$$\vec{v}(t) \cdot \vec{v}'(t) = 0.$$

Since $\vec{v}'(t) = \vec{a}(t)$, we have

$$\vec{v}(t) \cdot \vec{a}(t) = 0.$$

Thus, we conclude that $\vec{v}(t)$ is orthogonal to $\vec{a}(t)$ for all t.

Problem 12 A particle has an acceleration $\vec{a}(t) = t\vec{i} + t^2\vec{j} + e^{-t}\vec{k}$ and its velocity vector $\vec{v}(0)$ is $\vec{i} + \vec{j}$. Find the its velocity vector.
solution)

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t)dt \\ &= \left(\int tdt\right)\vec{i} + \left(\int t^2dt\right)\vec{j} + \left(\int e^{-t}dt\right)\vec{k} \\ &= \left(\frac{t^2}{2} + c_1\right)\vec{i} + \left(\frac{t^3}{3} + c_2\right)\vec{j} + (-e^{-t} + c_3)\vec{k}.\end{aligned}$$

Now,

$$\vec{v}(0) = c_1\vec{i} + c_2\vec{j} + (-1 + c_3)\vec{k} = \vec{i} + \vec{j}.$$

$$c_1 = 1, \quad c_2 = 1, \quad -1 + c_3 = 0.$$

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 1.$$

Hence, its velocity vector is

$$\vec{v}(t) = \left(\frac{t^2}{2} + 1\right)\vec{i} + \left(\frac{t^3}{3} + 1\right)\vec{j} + (-e^{-t} + 1)\vec{k}.$$

Problem 13 Let $\vec{r}(t) = 2\cos t^3\vec{i} + 2\sin t^3\vec{j}$.

a) Reparametrize the curve with respect to the arc length from the point where $t = 0$ in the direction of increasing t.

solution)

$$\vec{r}'(t) = -6t^2\sin t^3\vec{i} + 6t^2\cos t^3\vec{j}.$$

$$|\vec{r}'(t)| = \sqrt{36t^4\sin^2 t^3 + 36t^4\cos^2 t^3} = \sqrt{36t^4(\sin^2 t^2 + \cos^2 t^2)} = 6t^2.$$

The arc length function $s(t)$ is

$$s = \int_0^t |\vec{r}'(u)|du = \int_0^t 6u^2 du = 2t^3.$$

So,

$$t^3 = \frac{s}{2}.$$

Hence,

$$\vec{r}(s) = 2\cos\frac{s}{2}\vec{i} + 2\sin\frac{s}{2}\vec{j}.$$

b) Using $\vec{r}''(t) = (-12t\sin t^3 - 18t^4\cos t^3)\vec{i} + (12t\cos t^3 - 18t^4\sin t^3)\vec{j}$, find the tangential component of acceleration.

solution)

$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'(t)|} = -\sin t^3\vec{i} + \cos t^3\vec{j}.$$

$$\begin{aligned} a_T &= \vec{T}\cdot\vec{a} = \vec{T}\cdot\vec{r}'' \\ &= \left(-\sin t^3\vec{i} + \cos t^3\vec{j}\right)\cdot\left((-12t\sin t^3 - 18t^4\cos t^3)\vec{i} + (12t\cos t^3 - 18t^4\sin t^3)\vec{j}\right) \\ &= 12t. \end{aligned}$$

Problem 14 True of false: Explain your answer.

a) There are vectors $\vec{u}$ and $\vec{v}$ so that $\vec{u}\times\vec{v} = 5$.

solution) No, since the cross product results in a vector.

b)The cross product is associative, that is, $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$.

solution) It is not true. For example, $(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$. But, $\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$.

c) The binormal vector $\vec{B}(t)$ is perpendicular to the ocsulating plane.

solution) Yes. The osculating plane contains $\vec{T}(t)$ and $\vec{N}(t)$. So, $\vec{T} \times \vec{N}$ is perpendicular to the osculating plane and $\vec{B} = \vec{T} \times \vec{N}$.

Fromulae that you must remember (other than definitions, for example, the unit tangent vector, unit normal vector, etc.)

Section 12.1:

1. The distance formula between two points

Section 12.2:

1. The length of the vector

Section 12.3:

1. The angle formula for the dot product
2. The vector projection of $\vec{b}$ onto $\vec{a}$ ($\text{proj}_{\vec{a}}\vec{b}$)
3. The scalar projection of $\vec{b}$ onto $\vec{a}$ ($\text{comp}_{\vec{a}}\vec{b}$)
4. The angle formula for the cross product

Section 13.3:

1. The arc length
2. The arc length function s
3. Curvature: first two in the lecture note

Section 13.4:

1. The tangential and normal components of acceleration