

Practice Problems for Exam 2

Problem 1 Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = \frac{x}{y} \sin(xy^2)$.

solution) Using product rule,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{1}{y} \sin(xy^2) + \frac{x}{y} y^2 \cos(xy^2) = \frac{1}{y} \sin(xy^2) + xy \cos(xy^2), \\ \frac{\partial z}{\partial y} &= -\frac{x}{y^2} \sin(xy^2) + \frac{x}{y} 2xy \cos(xy^2) = -\frac{x}{y^2} \sin(xy^2) + 2x^2 \cos(xy^2).\end{aligned}$$

Problem 2 Find $f_x(1, 1)$ and $f_y(1, 1)$ given that $f(x, y) = y - e^{xy^2} + \sqrt{x^2 + 1}$.

solution)

$$\begin{aligned}f_x(x, y) &= -y^2 e^{xy^2} + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} 2x = -y^2 e^{xy^2} + x(x^2 + 1)^{-\frac{1}{2}}, \\ f_y(x, y) &= 1 - 2xye^{xy}.\end{aligned}$$

So,

$$f_x(1, 1) = -e + \frac{1}{\sqrt{2}}, \quad f_y(1, 1) = 1 - 2e.$$

Problem 3 Find the second derivatives of $f(x, y) = \sqrt{16 - 9x^2 - 4y^2}$.

solution)

$$\frac{\partial f}{\partial x} = \frac{1}{2}(16 - 9x^2 - 4y^2)^{-\frac{1}{2}}(-18x) = -9x(16 - 9x^2 - 4y^2)^{-\frac{1}{2}},$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(16 - 9x^2 - 4y^2)^{-\frac{1}{2}}(-8y) = -4y(16 - 9x^2 - 4y^2)^{-\frac{1}{2}},$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = -9(16 - 9x^2 - 4y^2)^{-\frac{1}{2}} + \frac{9}{2}x(-18x)(16 - 9x^2 - 4y^2)^{-\frac{3}{2}} \\ &= -9(16 - 9x^2 - 4y^2)^{-\frac{1}{2}} - 81x^2(16 - 9x^2 - 4y^2)^{-\frac{3}{2}} \\ &\quad \left(= 9(4y^2 - 16)(16 - 9x^2 - 4y^2)^{-\frac{3}{2}} \right), \end{aligned}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{9}{2}x(16 - 9x^2 - 4y^2)^{-\frac{3}{2}}(-8y) = -36xy(16 - 9x^2 - 4y^2)^{-\frac{3}{2}}.$$

Similarly,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = -36xy(16 - 9x^2 - 4y^2)^{-\frac{3}{2}}.$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -4(16 - 9x^2 - 4y^2)^{-\frac{1}{2}} - 4y(-8y)(16 - 9x^2 - 4y^2)^{-\frac{3}{2}} \\ &= -4(16 - 9x^2 - 4y^2)^{-\frac{1}{2}} - 16y^2(16 - 9x^2 - 4y^2)^{-\frac{3}{2}} \\ &\quad \left(= -4(16 - 9x^2)(16 - 9x^2 - 4y^2)^{-\frac{3}{2}} \right), \end{aligned}$$

Problem 4 Evaluate $\lim_{(x,y) \rightarrow (0,0)} \frac{1 + xy}{1 + x^2 + y^2}$.

solution) Since $(0, 0)$ is in the domain,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{1 + xy}{1 + x^2 + y^2} = \frac{1 + 0}{1 + 0 + 0} = 1.$$

Problem 5 Using polar coordinates, evaluate $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2}$.

solution)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos \theta \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} r \cos \theta \sin^2 \theta = 0.$$

Problem 6 Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$ does not exist.

solution) Along the line $y = mx$,

$$\lim_{x \rightarrow 0} \frac{mx^2}{x^2 + m^2x^2} = \lim_{x \rightarrow 0} \frac{mx^2}{(1 + m^2)x^2} = \lim_{x \rightarrow 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}.$$

Since the limit has different values as m varies, the limit does not exist.

Problem 7 Find the gradient vector at $(1, 1, 2)$ of $f(x, y, z) = \ln(x^2 + y^2 + z^2)$.

solution)

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= \langle f_x, f_y, f_z \rangle = \left\langle \frac{2x}{x^2 + y^2 + z^2}, \frac{2y}{x^2 + y^2 + z^2}, \frac{2z}{x^2 + y^2 + z^2} \right\rangle. \\ \vec{\nabla} f(1, 1, 2) &= \left\langle \frac{2}{6}, \frac{2}{6}, \frac{4}{6} \right\rangle = \left\langle \frac{1}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle.\end{aligned}$$

Problem 8 Find the directional derivative of $f(x, y) = 3xy^2 - 4x^3y$ in the direction of $3\vec{i} + 4\vec{j}$.

solution) The unit vector along the given direction is

$$\vec{u} = \frac{\langle 3, 4 \rangle}{\sqrt{9 + 16}} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \langle 3y^2 - 12x^2y, 6xy - 4x^3 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{9y^2 - 36x^2y + 24xy - 16x^3}{5}.$$

Problem 9 Find the directional derivative of $f(x, y, z) = x^2y + xy^2 + z^2$ at $(1, 1, 1)$ toward the point $(3, 1, 2)$.

solution) The vector from $(1, 1, 1)$ to $(3, 1, 2)$ is $\langle 3 - 1, 1 - 1, 2 - 1 \rangle = \langle 2, 0, 1 \rangle$. So the unit vector along this vector is

$$\frac{\langle 2, 0, 1 \rangle}{\sqrt{5}} = \left\langle \frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \right\rangle.$$

$$\vec{\nabla} f(x, y, z) = \langle 2xy + y^2, x^2 + 2xy, 2z \rangle.$$

So, $\vec{\nabla} f(1, 1, 1) = \langle 3, 3, 2 \rangle$. Hence,

$$D_{\vec{u}} f(1, 1, 1) = \vec{\nabla} f(1, 1, 1) \cdot \vec{u} = \langle 3, 3, 2 \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \right\rangle = \frac{8}{\sqrt{5}}.$$

Problem 10 The temperature at (x, y) is given by $T(x, y) = xy - x$. Find a vector in the direction in which the temperature increase most rapidly at $(1, 1)$ and find the maximum rate of increase in temperature at $(1, 1)$.

solution) Note that the gradient is the vector which points in the direction of the largest increase and its magnitude is the maximum rate of increase.

$$\vec{\nabla} T(x, y) = \langle y - 1, x \rangle.$$

$$\vec{\nabla} T(1, 1) = \langle 0, 1 \rangle.$$

So the temperature increase most rapidly at $(1, 1)$ in the direction of $\vec{\nabla} T(1, 1) = \langle 0, 1 \rangle$. And the maximum rate of increase is $|\vec{\nabla} T(1, 1)| = |\langle 0, 1 \rangle| = 1$.

Problem 11 Find the rate of change of $f(x, y, z) = x^2 + y^2 + z^2$ with respect to t along the curve $\vec{r}(t) = 2 \cos(10t)\vec{i} + 3 \sin(10t)\vec{j} - 20t\vec{k}$.

solution) We are looking for $\frac{df}{dt}$ when $f(x, y, z) = x^2 + y^2 + z^2$ and $x(t) = 2 \cos(10t)$, $y(t) = 3 \sin(10t)$, $z(t) = -20t$.
Using chain rule,

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \\ &= 2x(-20 \sin(10t)) + 2y(30 \cos(10t)) + 2z(-20) \\ &= -40x \sin(10t) + 60y \cos(10t) - 40z. \end{aligned}$$

Problem 12 Use the chain rule to find $\frac{\partial u}{\partial t}$ if $u = \sqrt{x^2 + y^2}$, $x = e^t$, and $y = \sin t$.

solution)

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \\ &= x(x^2 + y^2)^{-\frac{1}{2}} e^t + y(x^2 + y^2)^{-\frac{1}{2}} \cos t. \end{aligned}$$

Problem 13 Use the chain rule to find $\frac{\partial z}{\partial u}$ if $z = x^3 + xy + y^2$, $x = 2u + v$, and $y = u - 2v$.

solution)

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \\ &= (3x^2 + y)(2) + (x + 2y)(1) = 6x^2 + 4y + x.\end{aligned}$$

Problem 14 Verify that $z = f(x^3 - y^2)$ satisfies the equation $2y \frac{\partial z}{\partial x} + 3x^2 \frac{\partial z}{\partial y} = 0$.

solution) Let $u = x^3 - y^2$. Using chain rule,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{dz}{du} \frac{\partial u}{\partial x} = 3x^2 \frac{dz}{du}, \\ \frac{\partial z}{\partial y} &= \frac{dz}{du} \frac{\partial u}{\partial y} = -2y \frac{dz}{du}.\end{aligned}$$

Hence,

$$\begin{aligned}2y \frac{\partial z}{\partial x} + 3x^2 \frac{\partial z}{\partial y} &= 2y(3x^2) \frac{dz}{du} + 3x^2(-2y) \frac{dz}{du} \\ &= 6x^2y \frac{dz}{du} - 6x^2y \frac{dz}{du} = 0.\end{aligned}$$

Problem 15 Use the implicit differentiation theorem to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(\frac{1}{2}, -1, 2)$ if $e^{xz} + \ln(yz + 3) - y = 1 + e$.

solution)

$$F(x, y, z) = e^{xz} + \ln(yz + 3) - y.$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{ze^{xz}}{xe^{xz} + \frac{y}{yz+3}},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\frac{z}{yz+3} - 1}{xe^{xz} + \frac{y}{yz+3}}.$$

Hence,

$$\frac{\partial z}{\partial x}\left(\frac{1}{2}, -1, 2\right) = -\frac{2e}{\frac{1}{2}e - 1} = \frac{4e}{2 - e}.$$

$$\frac{\partial z}{\partial y}\left(\frac{1}{2}, -1, 2\right) = \frac{2}{2 - e}.$$

Problem 16 Find an equation for the tangent plane to the surface $4x^2 + 9y^2 + z = 17$ at the point $(-1, 1, 4)$.

solution) A normal vector to the tangent plane to the surface $F(x, y, z) = k$ (k a scalar) is $\vec{\nabla}F$.

$$\vec{\nabla}F(x, y, z) = \langle 8x, 18y, 1 \rangle, \quad \text{with } F(x, y, z) = 4x^2 + 9y^2 + z = 17.$$

So,

$$\vec{\nabla}F(-1, 1, 4) = \langle -8, 18, 1 \rangle.$$

Hence, the equation of the tangent plane at $(-1, 1, 4)$ is

$$-8(x + 1) + 18(y - 1) + (z - 4) = 0.$$

$$8x - 18y - z = -30.$$

Problem 17 Find a point on the surface $z = 16 - 4x^2 - y^2$ at which the tangent plane is perpendicular to the line $x = 3 + 4t$, $y = 2t$, $z = 2 - t$.

solution) A normal vector to the tangent plane at (x, y, z) is

$$\vec{\nabla}F(x, y, z) = \langle -8x, -2y, -1 \rangle,$$

where $F(x, y, z) = 16 - 4x^2 - y^2 - z = 0$. Also, the direction vector of the line is given by $\langle 4, 2, -1 \rangle$ by reading off the parametric equation of the line.

Since the tangent plane is perpendicular to the line, a normal vector to the tangent plane is parallel to the line. Hence,

$$\langle -8x, -2y, -1 \rangle = k \langle 4, 2, -1 \rangle, \quad k: \text{constant.}$$

$$-8x = 4k, \quad -2y = 2k, \quad -1 = -k.$$

$$k = 1, \quad -8x = 4, \quad -2y = 2.$$

$$x = -\frac{1}{2}, \quad y = -1.$$

And,

$$z = 16 - 4\left(-\frac{1}{2}\right)^2 - (-1)^2 = 14.$$

Hence, at the point $\left(-\frac{1}{2}, -1, 14\right)$, the tangent plane is perpendicular to the given line.

Problem 18 Find the differential df given that $f(x, y, z) = z^4 - 3yz^2 + y \sin z$.

solution)

$$df = f_x dx + f_y dy + f_z dz = 0dx + (3z^2 + \sin z)dy + (4z^3 - 6yz + y \cos z)dz.$$

Problem 19 Find the linear approximation of $f(x, y) = \sqrt{x^2 + y^2}$ at $(3, 4)$ and use it to estimate the number $\sqrt{(2.98)^2 + (4.01)^2}$.

solution)

$$f(3, 4) = 5, \quad f_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad f_x(3, 4) = \frac{3}{5}, \quad f_y = \frac{y}{\sqrt{x^2 + y^2}}, \quad f_y(3, 4) = \frac{4}{5}.$$

Linear approximation $L(x, y)$ at $(3, 4)$ is

$$L(x, y) = f(3, 4) + f_x(3, 4)(x-3) + f_y(3, 4)(y-4) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4).$$

So,

$$\begin{aligned} \sqrt{(2.98)^2 + (4.01)^2} &\approx 5 + \frac{3}{5}(2.98 - 3) + \frac{4}{5}(4.01 - 4) \\ &= 5 + \frac{3}{5}(-0.02) + \frac{4}{5}(0.01) = 5 + \frac{-0.06 + 0.04}{5} = 4.996. \end{aligned}$$

Problem 20 Find the critical point(s) of $f(x, y) = 3xy - 5x^2 - y^2 + 5x - 2y$ and classify the critical point(s).

solution)

$$\begin{cases} f_x = 3y - 10x + 5 = 0 \\ f_y = 3x - 2y - 2 = 0. \end{cases}$$

Solving for x and y , we get $x = \frac{4}{11}$ and $y = -\frac{5}{11}$.

$$f_{xx} = -10, \quad f_{xy} = 3, \quad f_{yy} = -2.$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 11.$$

And $f_{xx} = -10 < 0$. Hence, by the second derivative test, f has a local maximum $f\left(\frac{4}{11}, -\frac{5}{11}\right) = \frac{15}{11}$ at $\left(\frac{4}{11}, -\frac{5}{11}\right)$.

Problem 21 Find three positive numbers whose product is 64 and whose sum is as small as possible.

solution) We want to minimize $f(x, y, z) = x + y + z$ with the side condition $g(x, y, z) = xyz = 64$, $x > 0, y > 0, z > 0$.

I:

$$z = \frac{64}{xy}, \quad \text{so} \quad f = x + y + z = x + y + \frac{64}{xy}.$$

Critical points are where

$$\begin{aligned} f_x = 1 - \frac{64}{x^2y} = 0, \quad f_y = 1 - \frac{64}{xy^2} = 0. \\ x^2y = 64, \quad xy^2 = 64. \\ y = \frac{64}{x^2}, \quad xy^2 = 64. \end{aligned}$$

So,

$$\begin{aligned} x \frac{(64)^2}{x^4} = 64, \quad \frac{64}{x^3} = 1. \\ x^3 = 64, \quad x = \sqrt[3]{64} = 4. \end{aligned}$$

Hence,

$$y = \frac{64}{16} = 4, \quad z = \frac{64}{16} = 4.$$

Hence, the sum is minimum when $x = y = z = 4$.

II: Using Lagrange multiplier method,

$$\begin{cases} f_x = \lambda g_x, & f_y = \lambda g_y, & f_z = \lambda g_z, \\ xyz = 64. \end{cases}$$

$$\begin{cases} 1 = \lambda yz, & 1 = \lambda xz, & 1 = \lambda xy, \\ xyz = 64. \end{cases}$$

Multiply by x through $1 = \lambda yz$, we get $x = \lambda xyz$.

Multiply by y through $1 = \lambda xz$, we get $y = \lambda xyz$.

Multiply by z through $1 = \lambda xy$, we get $z = \lambda xyz$. Hence, we have

$$\begin{cases} x = \lambda xyz, & y = \lambda xyz, & z = \lambda xyz, \\ xyz = 64. \end{cases}$$

$$\begin{cases} x = y = z \\ xyz = 64. \end{cases}$$

So,

$$x^3 = 64, \quad x = 4.$$

Hence the sum is minimum when $x = y = z = 4$.

Problem 22 Use the Lagrange multiplier method to find the point on $z^2 = x^2 + y^2$ that is closest to the point $(2, 2, 0)$.

solution) Let (x, y, z) be the point on $z^2 = x^2 + y^2$. Then, the distance between (x, y, z) and the point $(2, 2, 0)$ is $d = \sqrt{(x-2)^2 + (y-2)^2 + z^2}$. Since d has minimum value iff d^2 has minimum value, we minimize $(x-2)^2 + (y-2)^2 + z^2$.

Now, minimize $f(x, y, z) = \sqrt{(x-2)^2 + (y-2)^2 + z^2}$ with the side condition $g(x, y, z) = x^2 + y^2 - z^2 = 0$.

$$f_x = 2(x-2), f_y = 2(y-2), f_z = 2z, g_x = 2x, g_y = 2y, \text{ and } g_z = -2z.$$

$$\begin{cases} 2(x-2) = 2\lambda x, & 2(y-2) = 2\lambda y, & 2z = -2\lambda z, \\ x^2 + y^2 - z^2 = 0. \end{cases}$$

So, we have

$$\begin{cases} x(1-\lambda) = 2, & y(1-\lambda) = 2, & z(1+\lambda) = 0 (\implies z = 0, \text{ or } \lambda = -1), \\ x^2 + y^2 - z^2 = 0. \end{cases}$$

Hence,

$$(1) \quad \begin{cases} x = \frac{2}{1-\lambda}, & y = \frac{2}{1-\lambda}, & z = 0, \\ x^2 + y^2 - z^2 = 0. \end{cases}$$

Or,

$$(2) \quad \begin{cases} x = \frac{2}{1-\lambda}, & y = \frac{2}{1-\lambda}, & \lambda = -1, \\ x^2 + y^2 - z^2 = 0. \end{cases}$$

But the case (1) is not possible, since

$$x^2 + y^2 - z^2 = \frac{4}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} - 0 \neq 0.$$

The case (2) becomes

$$\begin{cases} x = 1, & y = 1, \\ x^2 + y^2 - z^2 = 0 (\implies 1 + 1 = z^2 \implies z = \pm\sqrt{2}). \end{cases}$$

Thus, it has a minimum value at $(1, 1, \pm\sqrt{2})$, i.e., $(1, 1, \pm\sqrt{2})$ is the point on $z^2 = x^2 + y^2$ that is closest to $(2, 2, 0)$.