Practice Problems for Exam 3

Problem 1 Calculate the following double integrals.

a) $\int_0^1 \int_{x^2}^x (x^2 + y^2) dydx$

solution)

$$\int_0^1 \int_{x^2}^x (x^2 + y^2) dydx = \int_0^1 \left[x^2 y + \frac{1}{3} y^3 \right]_{y=x^2}^{x} dx$$

$$= \int_0^1 \left(x^3 + \frac{1}{3} x^3 - x^4 - \frac{1}{3} x^6 \right) dx = \left[\frac{1}{3} x^4 - \frac{1}{5} x^5 - \frac{1}{21} x^7 \right] = \frac{3}{35}.$$

b) $\int_0^1 \int_0^x e^{x^2} dydx$

solution)

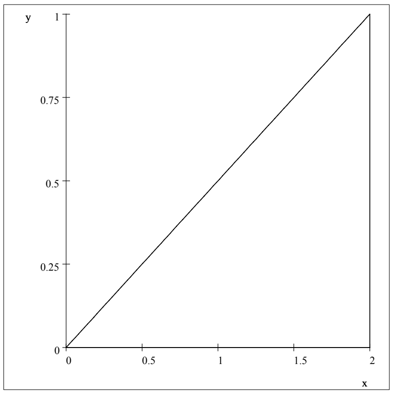
$$\int_0^1 \int_0^x e^{x^2} dydx = \int_0^1 \left[y e^{x^2} \right]_{y=0}^{x} dx$$

$$= \int_0^1 x e^{x^2} dx, \quad u = x^2,\ du = 2xdx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} e^u du = \left[\frac{1}{2} e^u \right]_{x=0}^{x=1} = \left[\frac{1}{2} e^{x^2} \right]_{x=0}^{1} = \frac{1}{2}(e - 1).$$

Problem 2 Evaluate $\int_0^1 \int_{2y}^2 \cos(x^2)dxdy$ by interchanging the order of integration.
solution) First, sketch the region of integration:

$$2y \le x \le 2, \quad 0 \le x \le 1.$$



$$\int_0^1 \int_{2y}^2 \cos(x^2)dxdy = \int_0^2 \int_0^{\frac{x}{2}} \cos(x^2)dydx$$

$$= \int_0^2 [y\cos(x^2)]_{y=0}^{\frac{x}{2}} dx = \int_0^2 \frac{x}{2}\cos(x^2)dx, \quad u = x^2,\ du = 2xdx$$

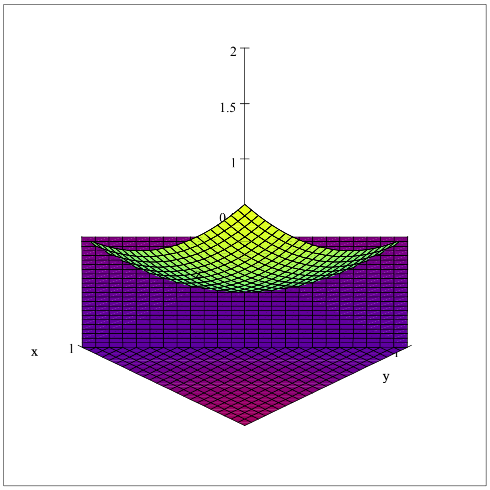
$$= \int_{x=0}^2 \frac{1}{4}\cos u du = \left[\frac{1}{4}\sin u\right]_{x=0}^2 = \left[\frac{1}{4}\sin x^2\right]_{x=0}^2 = \frac{1}{4}\sin 4.$$

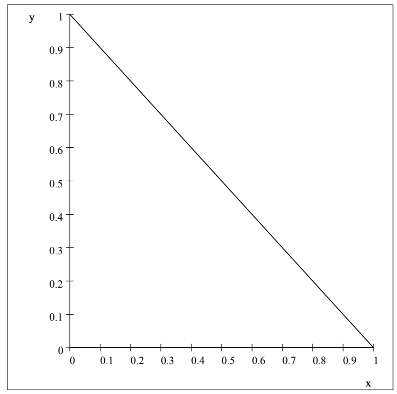
Problem 3 Use a double integral to find the area bounded by $y = x^2$ and $y = \sqrt{x}$.
solution) Let D be the region as described.

$$A(D) = \iint_D dA = \int_0^1 \int_{x^2}^{\sqrt{x}} dydx = \cdots = \frac{1}{3}.$$

Problem 4 Use a double integral to find the volume of the solid in the first octant bounded by the paraboloid $z = x^2 + y^2$ and the planes $z = 0, x + y = 1$.

solution) The given solid is the region under the paraboloid $z = x^2 + y^2$ and above the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.





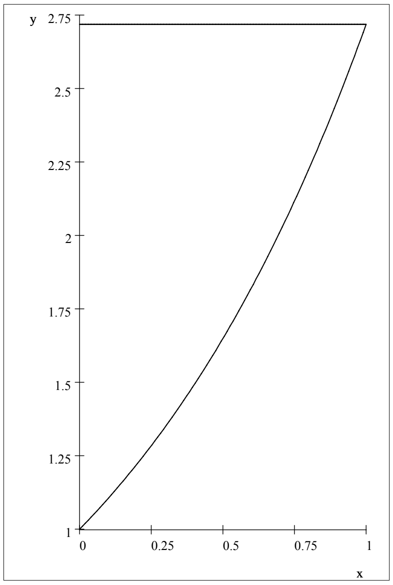
$$D: 0 \le y \le 1 - x, \quad 0 \le x \le 1.$$

So, the volume is

$$\iint_D (x^2 + y^2) dA = \int_0^1 \int_0^{1-x} (x^2 + y^2) dy dx = \cdots = \frac{1}{6}.$$

Problem 5 Calculate the average value of $f(x,y) = e^x$ over the region $D:\ 0 \le x \le \ln y,\ \ 1 \le y \le e.$

solution) Note that the average value $f_{\text{avg}} = \dfrac{1}{A(D)} \iint_D f(x,y)\,dA.$



$$A(D) = \iint_D dA = \int_1^e \int_0^{\ln y} dxdy = \int_0^1 \int_{e^x}^e dydx = \int_0^1 (e - e^x)dx = [ex - e^x]_{x=0}^1 = e - e - (0 - e^0) = 1.$$

Or,

$$A(D) = \iint_D dA = \int_1^e \int_0^{\ln y} dxdy = \int_1^e \ln y dy, \quad u = \ln y, dv = dy$$

$$= [y \ln y]_{y=1}^e - \int_1^e dy, \quad \text{integration by parts}$$

$$= e \ln e - \ln 1 - [y]_1^e = e - (e-1) = 1.$$

So,

$$f_{\text{avg}} = \int_0^1 \int_{e^x}^e e^x dydx = \int_0^1 [ye^x]_{y=e^x}^{y=e} dx = \int_0^1 (e^{x+1} - e^{2x})dx$$

$$= \left[e^{x+1} - \frac{1}{2} e^{2x} \right]_{x=0}^{x=1} = e^2 - \frac{1}{2}e^2 - \left(e - \frac{1}{2}\right) = \frac{1}{2}e^2 - e + \frac{1}{2}.$$

Or,

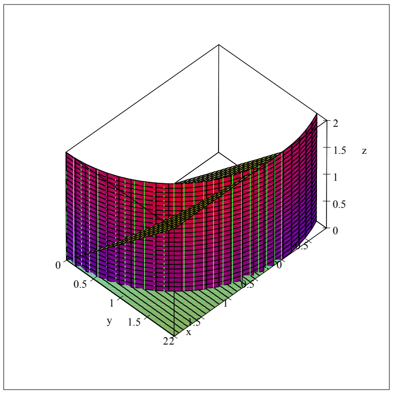
$$f_{\text{avg}} = \int_1^e \int_0^{\ln y} e^x dxdy = \int_1^e [e^x]_{x=0}^{\ln y} dy$$

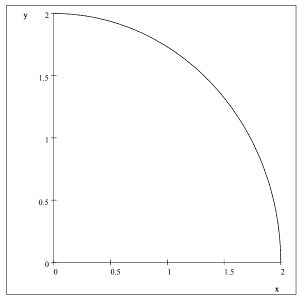
$$= \int_1^e (e^{\ln y} - 1)dy = \int_1^e (y-1)dy = \cdots = \frac{e^2}{2} - e + \frac{1}{2}.$$

Here, note that $e^{\ln y} = y$.

Problem 6 Use a triple integral to find the volume of the solid in the first octant bounded by the cylinder $x^2 + y^2 = 4$, and the planes $z = y$, $z = 0$.

solution) Let E be the region described as above. Then we know that the volume of E is $\iiint_E dV$. Project the region E onto the xy-plane and denote the projection by D.





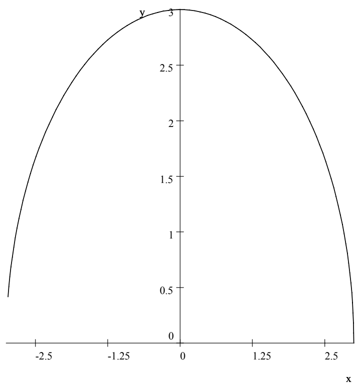
$$\iiint_E dV = \iint_D \int_0^y dz dA = \iint_D y dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 r \sin\theta r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \sin\theta \right]_{r=0}^{2} d\theta = \cdots = \frac{8}{3}.$$

Problem 7 Calculate $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{1}{\sqrt{x^2+y^2}} dxdy$ by changing to polar coordinates.

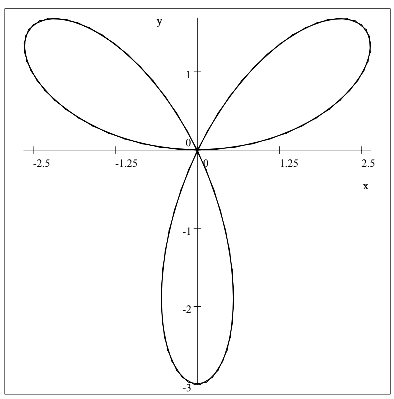
solution) $D : -\sqrt{9-y^2} \le x \le \sqrt{9-y^2}, \quad 0 \le y \le 3$



$$\int_0^{\pi}\int_0^3 \frac{1}{r} r dr d\theta = \int_0^{\pi}\int_0^3 dr d\theta = 3\pi.$$

Problem 8 Use a double integral in polar coordinates to find the area that is inside $r = 3\sin 3\theta$.

solution)

$r = 3\sin 3\theta$ 

First, find the area of one loop and multiply by 3.

$$3\sin 3\theta = 0.$$

$$3\theta = 0, \pi, 2\pi, \ldots, \quad \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots.$$

So, the area of one loop is

$$\int_0^{\frac{\pi}{3}}\int_0^{3\sin 3\theta} r dr d\theta = \int_0^{\frac{\pi}{3}} \left[\frac{r^2}{2}\right]_{r=0}^{3\sin 3\theta} d\theta = \int_0^{\frac{\pi}{3}} \frac{9}{2}\sin^2 3\theta d\theta$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{3}} \frac{1-\cos 6\theta}{2} d\theta, \quad \text{by the double angle formula } \sin^2 u = \frac{1-\cos 2u}{2}$$

$$= \frac{9}{4}\int_0^{\frac{\pi}{3}} (1-\cos 6\theta)d\theta = \cdots = \frac{3}{4}\pi.$$

Hence, the area that is inside $r = 3\sin(3\theta)$ is $3 \cdot \frac{3}{4}\pi = \frac{9}{4}\pi$.

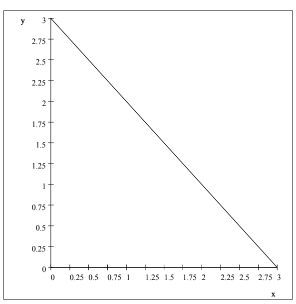
Problem 9 Evaluate $\int_0^1 \int_0^z \int_0^{\sqrt{yz}} x dx dy dz$.
solution)

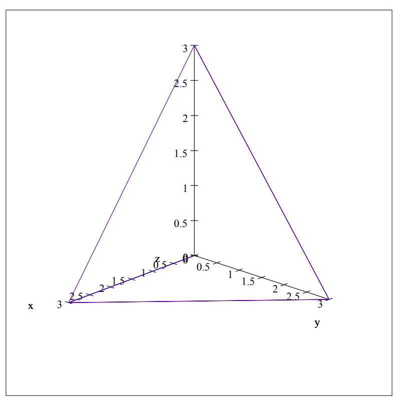
$$\int_0^1 \int_0^z \left[\frac{x^2}{2} \right]_{x=0}^{x=\sqrt{yz}} dydz$$

$$= \int_0^1 \int_0^z \frac{yz}{2} dydz = \int_0^1 \left[\frac{y^2 z}{4} \right]_{y=0}^{y=z} dz$$

$$= \int_0^1 \frac{z^3}{4} dz = \left[\frac{z^4}{16} \right]_{z=0}^{1} = \frac{1}{16}.$$

Problem 10 Set up an iterated integral for $\int\int\int_T x dV$, where T is the tetrahedron bounded by $x+y+z=3$ and the coordinate planes.
solution)

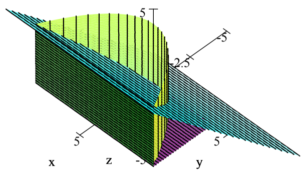


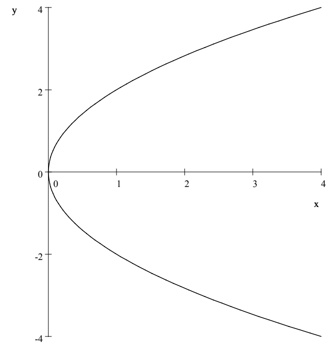


Project T onto xy-plane. We get a triangle with vertices $(0,0)$, $(3,0)$, and $(0,3)$.

$$\int_0^3 \int_0^{3-x} \int_0^{3-x-y} x dz dy dx.$$

Problem 11 Set up a triple integral for the volume of the solid bounded by the cylinder $y^2 = 4x$, and the planes $z = 0$, $z = x$, $x = 4$.
solution)

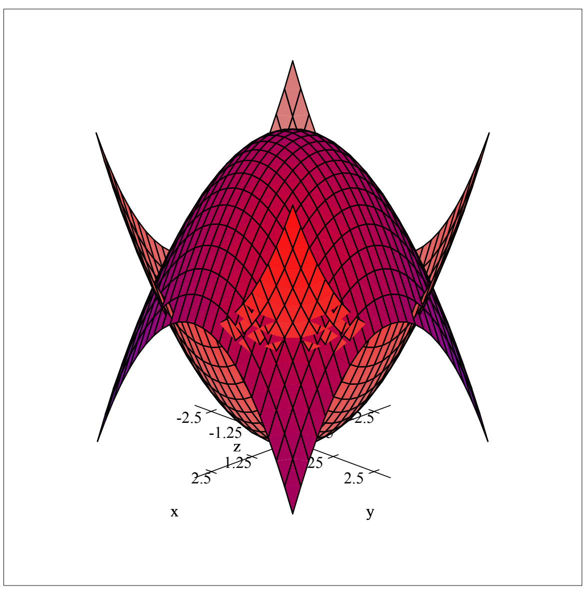


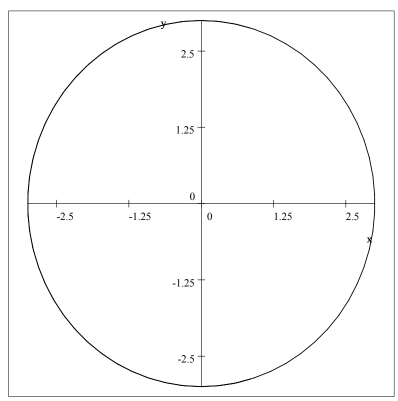


Let E be the given solid and project it onto xy-plane.
Then

$$V(E) = \iiint_E dV = \int_{-4}^{4} \int_{\frac{y^2}{4}}^{4} \int_0^x dzdxdy.$$

Problem 12 Set up a double integral for the volume of the solid enclosed by $z = x^2 + y^2$, $z = 18 - x^2 - y^2$ in polar coordinates.
solution)





Intersection of two surfaces is

$$x^2 + y^2 = 18 - x^2 - y^2.$$

$$x^2 + y^2 = 9.$$

If we project the given solid onto *xy*-plane, we have a disk

$$D: \; x^2 + y^2 = 9.$$

So, the volume of the solid is

$$\iint_D (18 - x^2 - y^2 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^3 (18 - 2r^2) r dr d\theta.$$

Problem 13 Find the center of mass of the tetrahedron T with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ if the density is proportional to the distance from the yz-plane.

solution) Note that the distance from the yz-plane is $|x|$. Since x-coordinates is positive in the given tetrahedron, the distance from the yz-plane is simply x. Hence, the density $\rho(x,y) = kx$, k : constant. By the formula, the center of mass $(\overline{x}, \overline{y}, \overline{z})$ is

$$\overline{x} = \frac{1}{m}\iiint_T x\rho(x,y)dV,$$

$$\overline{y} = \frac{1}{m}\iiint_T y\rho(x,y)dV,$$

$$\overline{z} = \frac{1}{m}\iiint_T z\rho(x,y)dV,$$

$$m = \iiint_T \rho(x,y)dV.$$

First,

$$m = \iiint_E kxdA = \int_0^1\int_0^{1-x}\int_0^{1-x-y} kxdzdydx = \cdots = \frac{k}{24}.$$

So,

$$\overline{x} = \frac{24}{k}\iiint_E xkxdV = 24\int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2dzdydx = \cdots = \frac{2}{5}.$$

$$\overline{y} = \frac{24}{k}\iiint_E ykxdV = 24\int_0^1\int_0^{1-x}\int_0^{1-x-y} xydzdydx = \cdots = \frac{1}{5}.$$

$$\overline{z} = \frac{24}{k}\iiint_E zkxdV = 24\int_0^1\int_0^{1-x}\int_0^{1-x-y} xzdzdydx = \cdots = \frac{1}{5}.$$

Problem 14 Find the surface ares of the part of the plane $2x+5y+z = 10$ that lies in the first octant.

solution) The domain D is the projection of this plane onto xy-plane. So, D is the triangle with vertices $(0,0)$, $(5,0)$, and $(0,2)$. Also, $z = f(x,y) = 10-2x-5y$ and $f_x = -2, f_y = -5$. By the formula, the surface area is

$$\int\int_D \sqrt{1+f_x^2+f_y^2}\, dA = \int\int_D \sqrt{1+4+25}\, dA$$

$$= \sqrt{30}\int\int_D dA = \sqrt{30}\,\text{Area}(D) = \sqrt{30}\,\frac{1}{2}\cdot 2\cdot 5 = 5\sqrt{30}.$$

Problem 15 Find the surface of the part of the sphere $x^2+y^2+z^2 = 4$ that lies within the cylinder $x^2+y^2 = 2x$ and above the xy-plane.

solution) Note that since the surface is above xy-plane, $z > 0$. Hence, we can rewrite the surface as $z = \sqrt{4-x^2-y^2}$.

$$f(x,y) = \sqrt{4-x^2-y^2}, f_x = \frac{-x}{\sqrt{4-x^2-y^2}}, f_y = \frac{-y}{\sqrt{4-x^2-y^2}}.$$

Since the surface lies within the cylinder $x^2+y^2 = 2x$, the domain $D :\ x^2+y^2 = 2x$. Writing D in polar coordinates, we have $r^2 = 2rcos\theta, \quad r = 2\cos\theta$.
By the formula, the surface area over D is

$$\int\int_D \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}\, dA$$

$$= \int\int_D \frac{2}{\sqrt{4-x^2-y^2}}\, dA, \quad \text{converting to polar coordinates}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} \frac{2}{\sqrt{4-r^2}} r dr d\theta, \quad u = 4-r^2,\ du = -2rdr$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{r=2\cos\theta} -\frac{1}{\sqrt{u}}du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [-2\sqrt{u}\,]_{r=0}^{r=2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-2\sqrt{4-r^2}\,\right]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-2\sqrt{4-4\cos^2\theta}+4\right)d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-2\cdot 2\sqrt{\sin^2\theta}+4\right)$$

$$= 2\int_0^{\frac{\pi}{2}} (-4\sin\theta+4)d\theta = \cdots = 4(\pi-2).$$

Note that $\sqrt{\sin^2\theta} = \begin{cases} \sin\theta, & \text{if } 0 \le \theta \le \frac{\pi}{2} \\ -\sin\theta, & \text{if } -\frac{\pi}{2} \le \theta \le 0. \end{cases}$ So,

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sin^2\theta}\; d\theta = 2\int_0^{\frac{\pi}{2}} \sin\theta d\theta.$