Review For the Final:

Problem 1 Find the general solutions of the following DEs.
a) $x^2y' - xy - y^2 = 0$
solution:

$$y' - \frac{y}{x} - \frac{y^2}{x^2} = 0 : \text{homogeneous equation.}$$

$$v = \frac{y}{x},\ y = vx, \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

$$v + x\frac{dv}{dx} = v + v^2.$$

$$x\frac{dv}{dx} = v^2 : \text{separable equation.}$$

$$\frac{dv}{v^2} = \frac{dx}{x}.$$

$$-\frac{1}{v} = \ln|x| + C.$$

$$-\frac{x}{y} = \ln|x| + C.$$

$$y = -\frac{x}{\ln|x| + C}.$$

Note: You can consider the above equation as a Bernoulli equation with $n = 2$.

b)$(x^2 - 1)y' + (x - 1)y = 1$
solution:

$$y' + \frac{x-1}{x^2-1}y = \frac{1}{x^2-1} : \text{linear equation.}$$

$$y' + \frac{y}{x+1} = \frac{1}{x^2-1}.$$

Integration factor $\rho(x) = e^{\int \frac{1}{x+1}dx} = e^{\ln(x+1)} = x + 1.$

$$\frac{d}{dx}(x+1)y = \frac{x+1}{x^2-1} = \frac{1}{x-1}.$$

$$(x+1)y = \ln|x-1| + C.$$

$$y = \frac{\ln|x-1| + C}{x+1}.$$

c)$y^2y' + 2xy^3 = 6x$

solution:

$$y' + 2xy = 6xy^{-2} : \text{Bernoulli equation with } n = -2.$$

$$v = y^{1-(-2)} = y^3,\ y = v^{1/3}\text{and} \quad \frac{dy}{dx} = \frac{1}{3}v^{-2/3}\frac{dv}{dx}.$$

$$\frac{1}{3}v^{-2/3}\frac{dv}{dx} + 2xv^{1/3} = 6xv^{-2/3}.$$

$$\frac{dv}{dx} + 6xv = 18x : \text{linear equation.}$$

Integrating factor $\rho(x) = e^{\int 6xdx} = e^{3x^2}$.

$$\frac{d}{dx}(e^{3x^2}v) = 18xe^{3x^2}.$$

$$e^{3x^2}v = 18\int xe^{3x^2}dx = 3e^{3x^2} + C,\ u = 3x^2\text{and } du = 6xdx.$$

$$v = 3 + Ce^{-3x^2}.$$

$$y = (3 + Ce^{-3x^2})^{1/3}.$$

d) $y' = \sqrt{x+y}$

sol) Let $v = x + y$, then $y = v - x$ and $\dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$.

$$\frac{dv}{dx} - 1\sqrt{v}.$$

$$\frac{dv}{dx} = \sqrt{v} + 1 : \text{separable equation.}$$

$$\frac{dv}{\sqrt{v}+1} = dx.$$

$$\int \frac{dv}{\sqrt{v}+1} = x + C.$$

Let $w = \sqrt{v} + 1$. Then

$$dx = \frac{1}{2} v^{-1/2} dv$$

and

$$dv = 2v^{1/2} dw = 2(w-1)dw.$$

Hence,

$$\int \frac{dv}{\sqrt{v}+1} = \int \frac{2(w-1)}{w} dw = \int (2 - \frac{2}{w}) dw$$
$$= 2w - 2\ln|w| + C = 2(\sqrt{v}+1) - 2\ln(\sqrt{v}+1) + C.$$

Thus,

$$x = 2(\sqrt{x+y}+1) - 2\ln(\sqrt{x+y}+1) + C.$$

Problem 2 A hemispherical bowl (with top radius 4) shaped water tank is slowly losing water at its lower end. As a result, the height of water in the tank, given by $y(t)$ satisfies

$$\frac{dy}{dt} = -\frac{1}{72}\frac{\sqrt{y}}{(8y - y^2)}.$$

a) Solve the DE for $y(t)$ when the tank is initially full.

solution: Since the tank is initially full and the top radius is 4, $y(0) = 4$.
By separating variables,

$$(8y^{1/2} - y^{3/2})dy = -\frac{1}{72}dt.$$

$$\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -\frac{t}{72} + C.$$

$$y(0) = 4, \quad C = \frac{448}{15}.$$

b) How long does it take for the tank to be empty?
solution: You need to find t for which $y(t) = 0$.

$$t = \frac{72 \cdot 148}{15}.$$

Problem 3 Determine whether the given functions are linearly independent or not.

a) $f(x) = e^x \sin x$, $g(x) = e^x \cos x$.

solution:

$$W(f,g) = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^{2x}(\sin x + \cos x) & e^x(\cos x - \sin x) \end{vmatrix}$$

$= e^{2x}(-\sin^2 x + \sin x \cos x) - e^{2x}(\sin x \cos x + \cos^2 x) = e^{-2x} \neq 0.$

Hence f and g are linearly independent.

b) $f(x) = \sin 2x$, $g(x) = \sin x \cos x$ and $h(x) = e^x$.

solution: Note that $\sin 2x = 2 \sin x \cos x$. Hence

$$f(x) - 2g(x) + 0h(x) = 0.$$

Since we can write 0 as a (nontrivial) linear combination of f, g and h, they are linearly dependent.

Remark : Nontrivial combination means linear combination other $0f + 0g + 0h = 0$.

Problem 4 Find the unique solution to the initial value problem

$$y'' - 7y' + 12y = (x+2)e^{3x}, \quad y(0) = 0, y'(0) = 2.$$

solution: Characteristic equation

$$r^2 - 7r + 12 = (r-3)(r-4) = 0, \quad r = 3, 4.$$

$$y_c = c_1 e^{3x} + c_2 e^{4x}.$$

Your first guess for y_p might be $y_p = (Ax + B)e^{3x}$. But since you have e^{3x} in y_c, the correct candidate should be

$$y_p = x(Ax + B)e^{3x}.$$

Hence,

$$y_p'' - 7y_p' + 12y_p = \left(-2Ax + (2A - B)\right) e^{3x} = (x+2)e^{3x}.$$

$$-2A = 1, \quad 2A - B = 1.$$

$$A = -\tfrac{1}{2}, \quad B = -3.$$

$$y = y_c + y_p = c_1 e^{3x} + c_2 e^{4x} - 3xe^{3x} - \tfrac{1}{2}x^2 e^{3x}.$$

Now,

$$y(0) = c_1 + c_2 = 0,$$

$$y'(0) = 3c_1 + 4c_2 - 3 = 2.$$

So, $c_1 = -5$ and $c_2 = 5$.

Hence, the unique solution y is

$$y = -5e^{3x} + 5e^{4x} - \tfrac{1}{2}x^2 e^{3x} - 3xe^{3x}.$$

Problem 5 Find the general solution of the differential equation $y''' - y'' + 4y' + 6y = e^{2x} + 2x$.

solution: Characteristic equation is $r^3 - r^2 + 4r + 6 = 0$. We observe that $r = -1$ is a solution and using this, we factor

$$r^3 - r^2 + 4r + 6 = (r+1)(r^2 - 2r + 6) = 0.$$

$$r = -1, \quad 1 \pm \sqrt{5}i.$$

So, $y_c = c_1 e^{-x} + c_2 e^x \cos\sqrt{5}x + c_3 e^x \sin\sqrt{5}x$. Find each particular solution for e^{2x}, $2x$.

For $\underline{e^{2x}}$: Try $y_{p_1} = Ae^{2x}$. Note that $y_{p_1}^{(n)} = A2^n e^{2x}$. So,

$$A(8 - 4 + 4 \cdot 2 + 6)e^{2x} = e^{2x}.$$

$$A = \tfrac{1}{18}.$$

$$y_{p_1} = \tfrac{1}{18}e^{2x}.$$

For $\underline{2x}$: Try $y_{p_2} = Bx + C$. Note that $y''_{p_2} = y'''(0) = 0$ and $y'_{p_2} = B$. So,

$$6Bx + (4B + 6C) = 2x.$$

$$6B = 2, \quad 4B + 6C = 0.$$

$$B = \tfrac{1}{3}, \quad C = -\tfrac{2}{9}.$$

$$y_{p_2} = \tfrac{1}{3}x - \tfrac{2}{9}.$$

Therefore,

$$y = y_c + y_{p_1} + y_{p_2} = c_1 e^{-x} + c_2 e^x \cos\sqrt{5}x + c_3 e^x \sin\sqrt{5}x + \tfrac{1}{18}e^{2x} + \tfrac{1}{3}x - \tfrac{2}{9}.$$

Problem 6 Find the general solution to the differential equation $y'' - 2y' + y = \frac{e^x}{1+x^2}$.

solution: Characteristic equation: $r^2 - 2r + 1 = 0, \quad (r-1)^2 = 0$, $r = 1, 1$. So, two linearly independent solutions are $y_1 = e^x$, $y_2 = xe^x$.

Since the right hand side is $f(x) = \frac{e^x}{1+x^2}$, we need to use the variation of parameter method.

$$y_p = u_1 y_1 + u_2 y_2 = u_1 e^x + u_2 x e^x,$$

where

$$u_1 = -\int \frac{y_2 f}{W}, \quad u_2 = \int \frac{y_1 f}{W}.$$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}(x+1) - xe^{2x} = e^{2x}.$$

$$u_1 = -\int \frac{xe^x \cdot \frac{e^x}{1+x^2}}{e^{2x}} dx = -\int \frac{x}{1+x^2} = -\frac{1}{2}\ln(1+x^2).$$

$$u_2 = \int \frac{e^x \cdot \frac{e^x}{1+x^2}}{e^{2x}} dx = \int \frac{1}{1+x^2} dx = \arctan x.$$

Hence,

$$y_p = -\frac{1}{2} e^x \ln(1+x^2) + xe^x \arctan x.$$

Thus, the general solution is

$$y = c_1 e^x + c_2 x e^x - \tfrac{1}{2} e^x \ln(1+x^2) + xe^x \arctan x.$$

Also do the problem 5 of the Exam 2.

Problem 7 Find the Fourier series solution of the end point problem

$$x'' + 2x = 1$$

$$x(0) = 0, \quad x(1) = 0.$$

solution: According to the boundary data, we need to find the sine series of 1, $0 < t < 1$.

$$b_n = 2\int_0^1 \sin n\pi t dt = \begin{cases} \frac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

So, $1 = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin n\pi t$. Now, we are looking for the solution $x(t)$ of

$$x'' + 2x = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin n\pi t.$$

Try

$$x(t) = \sum_{n \text{ odd}} B_n \sin n\pi t.$$

This would satisfy the end point conditions. We need to determine B_n.

$$x'' = -\sum_{n \text{ odd}} n^2\pi^2 \sin n\pi t.$$

Hence,

$$\sum_{n\text{odd}} B_n(2 - \pi^2 n^2) \sin n\pi t = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin n\pi t.$$

It follow that

$$B_n(2 - \pi^2 n^2) = \frac{4}{n\pi}, \quad \text{for } n \text{ odd.}$$

So, we have

$$B_n = \frac{4}{n\pi(2 - n^2\pi^2)}, \quad \text{for } n \text{ odd.}$$

Therefore,

$$x(t) = \sum_{n \text{ odd}} \frac{4}{n\pi(2 - n^2\pi^2)} \sin n\pi t.$$

Problem 8 Find a particular solution of the following equations.

a) $x'' + 2x = \sin t$

solution: Using undetermined coefficient method, set $x_p = A\sin t + B\cos t$. Note here that x_c does not have either $\sin t$ or $\cos t$. Since there is no x' term, we can try $x_p = A\sin t$. We can easily deduce that $A = 1$. So, $x_p = \sin t$.

b) $x'' + 2x = \sum_{n \text{ odd}} \frac{4}{n}\sin nt$. (Find a formal Fourier series solution.)

solution: Since there is no x' term, we can try

$$x_p(t) = \sum_{n \text{ odd}} B_n \sin nt.$$

Then,

$$x_p''(t) = \sum_{n \text{ odd}} -n^2 B_n \sin nt.$$

Plugging in x_p, x_p'' into the equation, we get

$$\sum_{n \text{ odd}} (-n^2 + 2)B_n \sin nt = \sum_{n \text{ odd}} \tfrac{4}{n} \sin nt.$$

So,

$$B_n = \frac{4}{n(2 - n^2)}.$$

Hence,

$$x_p(t) = \sum_{n \text{ odd}} \frac{4}{n(2 - n^2)} \sin nt.$$

Problem 9 Consider the following eigenvalue problem

$$X'' + \lambda X = 0, \quad 0 < x < \pi$$

$$X(0) = 0, \quad X'(\pi) = 0.$$

Show that the eigenvalues λ_n and eigenfunctions X_n are given by

$$\lambda_n = \frac{(2n-1)^2}{2^2}, \quad X_n = \sin\frac{(2n-1)x}{2}, n = 1, 2, \ldots.$$

You may use the following fact: $\cos x = 0$ if and only if $x = \frac{n\pi}{2}, n = 1, 3, 5, \ldots$ (i.e, n: odd).

solution: i) $\lambda = 0$: $X'' = 0$. $X(x) = C$. Since $X(0) = 0$, $C = 0$. Hence $\lambda = 0$ is not an eigenvalue.

ii) $\lambda < 0$: set $\lambda = -\alpha^2$, $\alpha > 0$ Then, $X'' + \alpha^2 X = 0$. The zeros of its characteristic equation is $\pm\alpha$. So,

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}.$$

$$X(0) = A + B = 0, \quad B = -A.$$

$$X'(\pi) = Ae^{\alpha\pi} + Be^{-\alpha\pi} = A(e^{\alpha\pi} - e^{-\alpha\pi}) = 0.$$

Since $e^{\alpha\pi} - e^{-\alpha\pi} \neq 0$, $A = 0$ and so $B = 0$. Hence, $X(x) = 0$. Thus, if $\lambda < 0$, it is not an eigenvalue.

iii) $\lambda > 0$: set $\lambda = \alpha^2$, $\alpha > 0$. Then, the general solution

$$X(x) = A\cos\alpha x + B\sin\alpha x.$$

$$X(0) = A = 0,$$

$$X'(\pi) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x = B\alpha\cos\alpha x = 0.$$

Now, we do not want $B = 0$. So, $\cos\alpha\pi = 0$, which implies that $\alpha\pi = \frac{2n-1}{2}, \quad n = 1, 2, \ldots.$ Thus the eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}$. The corresponding solutions (eigenfunctions) are $X_n(x) = \sin\frac{(2n-1)x}{2}$.

Problem 11 By separating the variables, solve the following wave equation

$$4u_{xx} = u_{tt}, \quad 0 < x < \pi, t > 0$$
$$u(0,t) = 0, \; u(\pi,t) = 0, t \geq 0.$$
$$u(x,0) = 1, \; u_t(x,0) = 0, \quad 0 < x < \pi.$$

solution: Set $u(x,t) = X(x)T(t)$.

$$4X''T = XT''.$$
$$\frac{X''}{X} = \frac{T''}{4T} = -\lambda.$$
$$X'' + \lambda X = 0.$$
$$X(0) = 0, \quad X(\pi) = 0.$$

So, $\lambda_n = \frac{n^2\pi^2}{\pi^2} = n^2$ and $X_n(x) = \sin nx$. Plug in $\lambda_n = n^2$ into $T'' + 4\lambda T = 0$. We get

$$T'' + 4n^2 = 0.$$
$$T(t) = c_1 \cos 2nt + c_2 \sin 2nt.$$

Since $u_t(x,0) = 0$, $T'(0) = 0$.

$$T'(t) = -2nc_1 \sin 2nt + 2nc_2 \cos 2nt, \quad T'(t) = 2nc_2 = 0, \quad c_2 = 0.$$

Hence,

$$T_n(t) = \cos 2nt.$$

Now

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos 2nt.$$
$$u(x,0) = \sum_{m=1}^{\infty} c_n \sin nx = 1.$$

Fourier sine series of 1 is

$$\sum_{n \text{ odd}} \tfrac{4}{n\pi} \sin nx.$$

Hence,

$$c_n = \frac{4}{n\pi}, \quad \text{if } n \text{ is odd and otherwise, } c_n = 0.$$

Thus,

$$u(x,t) = \sum_{n \text{ odd}} \tfrac{4}{n\pi} \sin nx \cos 2nt.$$

Problem 12 Find the solution of the following problem. You are not required to determine the coefficients.

$$\begin{gathered} u_{xx} + u_{yy} = 0, \quad 0 < x < 2, 0 < y < 2 \\ u(x,0) = u(x,2) = 0 \\ u(0,y) = 0. \end{gathered}$$

solution:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 2,\ 0 < y < 2 \tag{1}$$

$$u(x,0) = u(x,2) = 0 \tag{2}$$

Set $u(x,y) = X(x)Y(y)$. Then, (1) tells us that

$$\begin{gathered} X''Y + XY'' = 0. \\ -\frac{X''}{X} = \frac{Y''}{Y} = -\lambda. \end{gathered}$$

So, we have

$$Y'' + \lambda Y = 0, \tag{3}$$

$$X'' - \lambda X = 0. \tag{4}$$

From (2), we know $Y(0) = Y(2) = 0$. Using these boundary conditions and the equation (3) above, we have a familiar eigenvalue problem

$$\begin{gathered} Y'' + \lambda Y = 0 \\ Y(0) = Y(2) = 0. \end{gathered}$$

Eigenvalues: $\lambda_n = \frac{n^2\pi^2}{4}$
Eigenfunctions: $Y_n(y) = \sin\frac{n\pi y}{2}$.

We find from $u(0,y) = 0$ that $X(0) = 0$. Now plug in $\lambda_n = \frac{n^2\pi^2}{4}$ into the equation (4) and consider

$$\begin{gathered} X'' - \tfrac{n^2\pi^2}{x} = 0 \\ X(0) = 0. \end{gathered}$$

We know that

$$X(x) = c_1 e^{\frac{n\pi x}{2}} + c_2 e^{-\frac{n\pi x}{2}}.$$

Since $X(0) = c_1 + c_2 = 0$, $c_2 = -c_1$.

Hence,

$$X(x) = c_1\left(e^{\frac{n\pi x}{2}} - e^{-\frac{n\pi x}{2}}\right).$$

Thus,

$$X_n(x) = e^{\frac{n\pi x}{2}} - e^{-\frac{n\pi x}{2}}.$$

So,

$$u(x,y)=\sum_{n=1}^{\infty}c_nX_n(x)Y_n(y)=\sum_{n=1}^{\infty}c_n\left(e^{\frac{n\pi x}{2}}-e^{-\frac{n\pi x}{2}}\right)\sin\frac{n\pi y}{2}.$$