Practice problems for section 9.7

Problem 1 Find the solution of the following problem.

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 2,\ 0 < y < 2 \tag{1}$$

$$u(x,0) = u(x,2) = 0 \tag{2}$$

$$u(0,y) = 0, \quad u(2,y) = \sin\frac{\pi y}{2}.$$

solution: Set $u(x,y) = X(x)Y(y)$. Then, (1) tells us that

$$X''Y + XY'' = 0.$$

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda.$$

So, we have

$$Y'' + \lambda Y = 0, \tag{3}$$

$$X'' - \lambda X = 0. \tag{4}$$

From (2), we know $Y(0) = Y(2) = 0$. Using these boundary conditions and the equation (3) above, we have a familiar eigenvalue problem

$$Y'' + \lambda Y = 0$$

$$Y(0) = Y(2) = 0.$$

Eigenvalues: $\lambda_n = \frac{n^2\pi^2}{4}$
Eigenfunctions: $Y_n(y) = \sin\frac{n\pi y}{2}$.

We find from $u(0,y) = 0$ that $X(0) = 0$. Now plug in $\lambda_n = \frac{n^2\pi^2}{4}$ into the equation (4) and consider

$$X'' - \frac{n^2\pi^2}{x} = 0$$

$$X(0) = 0.$$

We know that

$$X(x) = c_1 e^{\frac{n\pi x}{2}} + c_2 e^{-\frac{n\pi x}{2}}.$$

Since $X(0) = c_1 + c_2 = 0$, $c_2 = -c_1$.
Hence,

$$X(x) = c_1\left(e^{\frac{n\pi x}{2}} - e^{-\frac{n\pi x}{2}}\right).$$

Thus,

$$X_n(x) = e^{\frac{n\pi x}{2}} - e^{-\frac{n\pi x}{2}}.$$

So,

$$u(x,y) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} \left(e^{\frac{n\pi x}{2}} - e^{-\frac{n\pi x}{2}}\right) \sin \frac{n\pi y}{2}.$$

Now let's determine c_n by using the nonhomogeneous condition $u(2,y) = \sin \frac{\pi y}{2}$.

$$u(2,y) = \sum_{n=1}^{\infty} c_n (e^{n\pi} - e^{-n\pi}) \sin \tfrac{n\pi y}{2} = \sin \tfrac{\pi y}{2}.$$

The right hand side is already a sine series, so we deduce that

$$c_1(e^{\pi} - e^{-\pi}) = 1, \quad c_n(e^{n\pi} - e^{-n\pi}) = 0 \quad \text{if } n \neq 1.$$

Hence,

$$c_1 = \frac{1}{e^{\pi} - e^{-\pi}} \quad \text{and} \quad c_n = 0, \quad \text{otherwise.}$$

Therefore,

$$u(x,t) = \frac{1}{e^{\pi} - e^{-\pi}} \left(e^{\frac{\pi x}{2}} - e^{-\frac{\pi x}{2}}\right) \sin \tfrac{\pi y}{2} \left(= \frac{\sinh \frac{x}{2} \sin \frac{\pi y}{2}}{\sinh \pi} \right).$$

Problem 2 Find the solution of the following problem.

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 2,\ 0 < y < 2$$
$$u(x,0) = u(x,2) = 0$$
$$u(0,y) = 1, \quad u(2,y) = 0.$$

solution: Set $u(x,y) = X(x)Y(y)$ and proceed as the problem 1. Then, we have

$$Y'' + \lambda Y = 0$$
$$Y(0) = 0, \quad Y(2) = 0.$$

Hence, $\lambda_n = \frac{n^2\pi^2}{4}$ and $Y_n(y) = \sin\frac{n\pi y}{2}$. As in the problem 1, the equation for X is

$$X'' - \lambda X = 0.$$

The general solution is as before

$$X(x) = c_1 e^{\frac{n\pi x}{2}} + c_2 e^{-\frac{n\pi x}{2}}.$$

But a homogeneous condition $u(2,y) = 0$ tells us that $X(2) = 0$. So,

$$X(2) = c_1 e^{n\pi} + c_2 e^{-n\pi} = 0.$$
$$c_2 e^{-n\pi} = -c_1 e^{n\pi}.$$

Multiply through by $e^{n\pi}$. Then,

$$c_2 = -c_1 e^{2\pi n}.$$

Hence, $X(x) = c_1 \left(e^{\frac{n\pi x}{2}} - e^{2\pi n} e^{-\frac{n\pi x}{2}}\right)$ and

$$X_n(t) = e^{\frac{n\pi x}{2}} - e^{2\pi n} e^{-\frac{n\pi x}{2}}.$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} c_n \left(e^{\frac{n\pi x}{2}} - e^{2\pi n} e^{-\frac{n\pi x}{2}}\right) \sin\frac{n\pi y}{2}.$$

Now using the nonhomogeneous condition $u(0,y) = 1$, determine c_n.

$$u(0,y) = \sum_{n=1}^{\infty} c_n (1 - e^{2n\pi}) \sin\tfrac{n\pi}{2} = 1.$$

Fourier sine series of 1 is

$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin\tfrac{n\pi y}{2}.$$

Hence,

$$c_n(1 - e^{2n\pi}) = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus,

$$u(x,y) = \sum_{n \text{ odd}} \frac{4}{n\pi(1-e^{2n\pi})} \left(e^{\frac{n\pi x}{2}} - e^{2\pi n} e^{-\frac{n\pi x}{2}}\right) \sin\frac{n\pi y}{2}.$$

The above answer is good enough. But let us try to write the answer in terms of sinh function.

Multiplying top and bottom by $e^{-n\pi}$, we have

$$\frac{e^{\frac{n\pi x}{2}} - e^{2\pi n} e^{-\frac{n\pi x}{2}}}{1-e^{2n\pi}} = \frac{e^{-n\pi} e^{\frac{n\pi x}{2}} - e^{n\pi} e^{-\frac{n\pi x}{2}}}{e^{-n\pi} - e^{n\pi}},$$

Multiplying top and bottom by -1 and combining the exponential functions, the above function becomes

$$\frac{e^{\frac{2n\pi - n\pi x}{2}} - e^{-(\frac{2n\pi - n\pi x}{2})}}{e^{n\pi} - e^{-n\pi}}$$

$$= \frac{\sinh(\frac{n\pi(2-x)}{2})}{\sinh n\pi}, \quad \text{by the definition of } \sinh x = \frac{e^x - e^{-x}}{2}.$$

Thus,

$$u(x,t) = \sum_{n \text{ odd}} \frac{4}{n\pi} \frac{\sinh\frac{n\pi(2-x)}{2}}{\sinh n\pi} \sin\tfrac{n\pi y}{2}.$$