Practice Problem for Sec 9-7: Polar Coordinates

Solve the following Dirichlet problem for the semicircular disk of radius $3 : x^2 + y^2 \leq 9$, $0 \leq \theta \leq \pi$.

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad (1)$$
$$u_\theta(r,0) = u_\theta(r,\pi) = 0, \quad (2)$$
$$u(3,\theta) = f(\theta) \qquad (3).$$

Assume that the solution $u(r,\theta)$ is bounded, and you may aslo assume that the eigenvalues and the eigenfunctions for $x'' + \lambda x = 0,\ x'(0) = x'(L) = 0$ are

$$\lambda_n = \frac{n^2\pi^2}{L^2},\ n = 0,1,2,\ldots,$$
$$x_n = \cos\frac{n\pi t}{L},\ n = 0,1,2,\ldots.$$

solution) Let $u(r,\theta) = R(r)T(\theta)$. Then, (1) becomes

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0$$
$$\left(R'' + \frac{1}{r}R'\right)T = -\frac{1}{r^2}RT''$$
$$\frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = \lambda.$$

Hence, we have

$$T'' + \lambda T = 0, \quad (4)$$
$$r^2R'' + rR' - \lambda R = 0 \quad (5).$$

And (2) becomes $T'(0) = T'(\pi) = 0$.

Using the given fact about the eigenvalues and eigenfunctions with $L = \pi$, we know that the differential equation (4) with the endpoints conditions (6) has a nontrivial solution $T_n(\theta) = \cos n\theta$ for $\lambda_n = n^2,\ n = 0,1,2,\ldots.$

$$T_n(\theta) = \cos n\theta \quad \text{for } \lambda_n = n^2, n = 0,1,2,$$

Now solve (5) for $\lambda_0 = 0$ and $\lambda_n = n^2\ (n = 1,2,\ldots)$ separately:
$\lambda_0 = 0\ \ (T_0 = 1):\ \ r^2R'' + rR' = 0.$ Use a substitution $v = R'$. Then,

$$r^2\frac{dv}{dr} + rv = 0.$$
$$\frac{1}{v}dv = -\frac{1}{r}dr$$
$$\ln|v| = -\ln r + C$$
$$v = \frac{C}{r}.$$
$$R' = \frac{C}{r} \Rightarrow R = \int \frac{C}{r}dr = C\ln r + D.$$

Since $u(r,\theta)$ is bounded and $\lim_{r\to 0}\ln r = -\infty$, we see that $C = 0$.

Hence, $R(r) = D$. Take $R_0 = 1$. Thus, $\boxed{u_0 = R_0 T_0 = 1.}$

$\lambda_n = n^2$ $(n = 1,2,\ldots)$ $(T_n(\theta) = \cos n\theta)$:

$$r^2 R'' + rR' - n^2 R = 0 : \text{ Euler equation}$$

Assume that $R(r) = r^\alpha$. Then,

$$(\alpha(\alpha-1) + \alpha - n^2)r^\alpha = 0.$$

$$\alpha(\alpha-1) + \alpha - n^2 = 0.$$

$$\alpha = \pm n.$$

Hence,

$$R(r) = c_1 r^n + c_2 r^{-n}.$$

Since $u(r,\theta)$ is bounded and $\lim_{r\to 0} r^{-n} = \infty$, $c_2 = 0$.

Hence, $R(r) = c_1 r^n$. Take $R_n = r^n$. Thus, $\boxed{u_n = R_n T_n = r^n \cos n\theta,\ n = 1,2,\ldots.}$

Therefore, the solution satisfying (1) and (2) is

$\boxed{u(r,\theta) = c_0 u_0 + \sum_{n=1}^\infty c_n u_n = c_0 + \sum_{n=1}^\infty c_n r^n \cos n\theta.}$

Now, determine $c_0, c_1, c_2, \ldots$ so that $c_0 + \sum_{n=1}^\infty c_n r^n \cos n\theta$ satisfies the condition (3).

$$c_0 + \sum_{n=1}^\infty c_n 3^n \cos n\theta = f(\theta).$$

$$c_0 = \frac{a_0}{2} = \frac{1}{2}\frac{2}{\pi}\int_0^\pi f(\theta)d\theta = \frac{1}{\pi}\int_0^\pi f(\theta)d\theta$$

$$c_n 3^n = a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta d\theta \Rightarrow c_n = \frac{2}{3^n \pi}\int_0^\pi f(\theta)\cos n\theta d\theta,\ n = 1,2,\ldots.$$