Math 385 B1 Exam 1
Name (Please Print.):________Key________________________________

Problem 1

(a) (2 points) Determine the order of $\frac{dy}{dx} + xy^2 = 0$, and also decide if the equation is linear or nonlinear.
solution) order is 1, and it is nonlinear.

(b) (4 points) Explain why the initial value problem $\frac{dy}{dx} = \sin(xy),\ y(0) = 0$ has a unique solution on some interval containing 0. Also find its solution by observation.
solution) $f(x,y) = \sin(xy)$, and $\frac{\partial f}{\partial y} = x\cos(xy)$ are continuous near $(0,0)$. Hence, by the existence and the uniqueness theorem, the given problem has a unique solution.
Consider the constant function $y \equiv 0$. Then, $\frac{dy}{dx} = 0$ and $\sin(xy) = \sin 0 = 0$.
Hence, the solution is $y \equiv 0$.

(c) (3 points) Consider the initial value problem $\frac{dy}{dx} = \sin(xy),\ y(\frac{\pi}{2}) = 1$. Find the slope of the solution at $x = \frac{\pi}{2}$.
solution) slope at $x = \frac{\pi}{2}$ is $\sin(\frac{\pi}{2} \cdot 1) = 1$.

(d) (10 points) Determine if the differential equation $(e^y + y\cos x) + (xe^y + \sin x)\frac{dy}{dx} = 0$ is exact. If so, find the solution satisfying the initial condition $y(\frac{\pi}{2}) = 0$.
solution) Rewriting the equation as

$$(e^y + y\cos x)dx + (xe^y + \sin x)dy = 0.$$

Let $M = e^y + y\cos x$ and $N = xe^y + \sin x$.
Then, $M_y = e^y + \cos x$ and $N_x = e^y + \cos x$. So, the equation is exact.
Hence there is a function $F(x,y)$ such that $F_x = e^y + y\cos x$ and $F_y = xe^y + \sin x$, and the general solution curves are given by $F(x,y) = C$.

$$F(x,y) = \int (e^y + y\cos x)dx + g(y) = xe^y + y\sin x + g(y)$$

$$\frac{\partial F}{\partial y} = xe^y + \sin x + \frac{dg}{dy} = xe^y + \sin x \Rightarrow \frac{dg}{dy} = 0 \Rightarrow g(y) = D.$$

Thus, a general solution is $xe^y + y\sin x + D = C \Rightarrow xe^y + y\sin x = E$.
$y(\frac{\pi}{2}) = 0;\ \frac{\pi}{2}e^0 + 0\sin\frac{\pi}{2} = E \Rightarrow E = \frac{\pi}{2}$.
Hence, the solution is $xe^y + y\sin x = \frac{\pi}{2}$.

Problem 2 (14 points each) Find a general solution of the following differential equations. All the primes denote with respect to the variable x.

(a) $yy' + y^2 = e^x$

solution)

$$y' + y = e^x y^{-1} \text{ : Bernoulli equation with } n = -1.$$

Use substitution $v = y^{1-n} = y^2 \Rightarrow y = v^{\frac{1}{2}},\ \frac{dy}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx}$.

$$\frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx} + v^{\frac{1}{2}} = e^x v^{-\frac{1}{2}}.$$

$$\frac{dv}{dx} + 2v = 2e^x \text{ : linear first order equation}$$

Integrating factor $\rho(x) = e^{\int 2dx} = e^{2x}$.

$$\frac{d}{dx}(e^{2x}v) = 2e^x e^{2x} = 2e^{3x}.$$

$$e^{2x}v = \int 2e^{3x}dx = \frac{2}{3}e^{3x} + C.$$

$$v = \frac{2}{3}e^x + Ce^{-2x}.$$

$$y = \left(\frac{2}{3}e^x + Ce^{-2x}\right)^{\frac{1}{2}}.$$

(b) $y' = \frac{xy + y^2}{x^2}$

solution)

$$y' = \frac{xy + y^2}{x^2} = \frac{y}{x} + \frac{y^2}{x^2}$$: homogeneous equation.

Let $v = \frac{y}{x}$. Then, $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

$$x\frac{dv}{dx} + v = v + v^2.$$

$$x\frac{dv}{dx} = v^2$$: separable

$$\int \frac{1}{v^2}dv = \int \frac{1}{x}dx.$$

$$-\frac{1}{v} = \ln|x| + C.$$

$$v = -\frac{1}{\ln|x| + C}.$$

$$y = -\frac{x}{\ln|x| + C.}$$

Problem 3 Consider a deer population $P(t)$ given by the following model
$\frac{dP}{dt} = -(P-1)(P-8),\ P(0) = 6.$
(a) (2 points) Find the critical points.
solution) $-(P-1)(P-8) = 0 \Rightarrow P = 1, 8.$

(b) (2 points) Use phase diagram to determine each critical points is stable or unstable.
solution) 8 : stable, 1 : unstable.

(c) (2 points) What can we expect as time goes?
solution) Population reaches 8 as time goes.

(d) (12 points) Solve explicitly for $P(t)$.
solution)

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\begin{aligned}
\frac{dP}{dt} &= -(P-1)(P-8). \\
\frac{1}{(P-1)(P-8)}dP &= -dt. \\
\frac{1}{7}\int\left(\frac{1}{P-8} - \frac{1}{P-1}\right)dP &= -\int dt. \\
\frac{1}{7}\ln\left|\frac{P-8}{P-1}\right| &= -t + C. \\
\ln\left|\frac{P-8}{P-1}\right| &= -7t + C. \\
\frac{P-8}{P-1} &= Ce^{-7t}. \\
(P-8) &= Ce^{-7t}P - Ce^{-7t}. \\
P(1 - Ce^{-7t}) &= 8 - Ce^{-7t}. \\
P &= \frac{8 - Ce^{-7t}}{1 - Ce^{-7t}}. \\
P(0) &= 6;\ \frac{8-C}{1-C} = 6 \Rightarrow C = -\frac{2}{5}. \\
P(t) &= \frac{8 + \frac{2}{5}e^{-7t}}{1 + \frac{2}{5}e^{-7t}} = \frac{40 + 2e^{-7t}}{5 + 2e^{-7t}}\left(= \frac{40e^{7t} + 2}{5e^{7t} + 2}\right).
\end{aligned}
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