Practice Problems: Chapter 1, Section 2.1 and 2.2:

Problem 1 Determine the order of the given differential equation, and determine if the equation is linear or nonlinear.

(a) $(1+y)^2 \frac{dy^2}{dx^2} + x\frac{dy}{dx} + y = e^x$.

solution) The order is 2. It is nonlinear, since it has $(1+y)^2$.

(b) $\frac{d^3y}{dx^3} + x\frac{dy}{dx} + (\cos^2 x)y = x^3$.

solution) The order is 3. It is linear.

Problem 2 Verify that the given function or functions is a solution of the differential equation.

(a) $xy' - y = x^2$; $y = 3x + x^2$.

solution)

$$y = 3x + x^2, \quad y' = 3 + 2x.$$

$$xy' - y = x(3+2x) - (3x + x^2) = x^2.$$

Since $y = 3x + x^2$ satisfies the given equation, it is a solution.

(b) $2x^2y'' + 3xy' - y = 0,\ x > 0;\ y_1(x) = x^{\frac{1}{2}},\ \ y_2(x) = x^{-1}$.

solution)

$$y_1(x) = x^{\frac{1}{2}}, \quad y_1{}'(x) = \frac{1}{2}x^{-\frac{1}{2}}, \quad y_1{}''(x) = -\frac{1}{4}x^{-\frac{3}{2}}.$$

$$\begin{aligned} 2x^2y'' + 3xy' - y &= 2x^2\left(-\frac{1}{4}x^{-\frac{3}{2}}\right) + 3x\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - x^{\frac{1}{2}} \\ &= -\frac{1}{2}x^{\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}} - x^{\frac{1}{2}} = 0. \end{aligned}$$

Since $y_1(x) = x^{\frac{1}{2}}$ satisfies the given equation, it is a solution.

Similarly, we can verify that $y_2(x) = x^{-1}$ is a solution.

Problem 3 Decide if the statement is true or false.

(a) The initial value problem $x\frac{dy}{dx} = 2y,\ y(1) = 0$ has a unique solution on some interval containing 1.

solution) Answer: True

Rewrite as $\frac{dy}{dx} = \frac{2y}{x}$.

Since both $f(x,y) = \frac{2y}{x}$ and $\frac{\partial f}{\partial y} = \frac{2}{x}$ are continuous near $(1,0)$, the given initial value problem has a unique solution.

(b) The initial value problem $x\frac{dy}{dx} = 2y,\ y(0) = 1$ has infinitely many solutions.

solution) Answer: False.

Note that $f(x,y) = \frac{2y}{x}$ is not continuous near $(0,1)$. Hence, the existence and the uniqueness theorem is not applicable here.

But using initial condition, we see that $0 \cdot \frac{dy}{dx} = 2$, which leads to $0 = 2$. (can not be true.)

Hence, there is no such a solution satisfying the given initial value condition.

Problem 4

(a) Consider the initial value problem $y' = \sqrt{y},\ y(0) = 1$. Find the slope of the solution at $x = 0$.

solution) Note that for a differential equation $y' = f(x,y)$, the slope of a solution, i.e., the slope of the tangnet line to a solution curve, at each point (x,y) is given by $f(x,y)$.

Hence, the slope at $x = 0$ is $\sqrt{y(0)} = \sqrt{1} = 1$.

(b) Consier the initial value problem $y' = e^{x+y},\ y(0) = 1$. Find the slope of the solution at $x = 0$.

solution)

$$e^{0+y(0)} = e^{0+1} = e.$$

Problem 5 Find a general solution of the following diffierential equations. All the primes denote with respect to the variable x.

(a) $y' + 2y = e^x$

solution)

$$y' + 2y = e^x \text{ : first linear equation}$$

$$\text{Integrating factor } \rho(x) = e^{\int 2dx} = e^{2x}.$$

Multiply through by e^{2x}, we get

$$\frac{d}{dx}(e^{2x}y) = e^{3x}.$$

$$e^{2x}y = \int e^{3x}dx = \frac{1}{3}e^{3x} + C.$$

$$y = \frac{1}{3}e^x + \frac{C}{e^{2x}}.$$

(b) $(1-y^2)y' - x^2 = 0.$

solution) Rewrite it as

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \text{ : separable}$$

$$(1-y^2)dy = x^2dx.$$

$$\int(1-y^2)dy = \int x^2dx.$$

$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C \left(\text{or, } 3y - y^3 = x^3 + C\right).$$

(c) $x^2y' - xy^2 - 3y^2 = 0$

solution)

$$x^2y' - xy^2 - 3y^2 = 0.$$

$$\frac{dy}{dx} = \frac{x+3}{x^2}y^2 \ : \ \text{separable}$$

$$\frac{1}{y^2}dy = \frac{x+3}{x^2}dx.$$

$$\int \frac{1}{y^2}dy = \int \frac{x+3}{x^2}dx = \int\left(\frac{1}{x} + \frac{3}{x^2}\right)dx.$$

$$-\frac{1}{y} = \ln|x| - \frac{3}{x} + C.$$

$$y = -\frac{1}{\ln|x| - \frac{3}{x} + C} = -\frac{x}{x\ln|x| - 3 + Cx}.$$

(d) $y' = \dfrac{x^2 + xy + y^2}{x^2}$

solution)

$$y' = \frac{x^2 + xy + y^2}{x^2} = 1 + \frac{y}{x} + \frac{y^2}{x^2} \ : \ \text{homogeneous equation}$$

Let $v = \frac{y}{x}$. Then, $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

So, the equation becomes

$$x\frac{dv}{dx} + v = 1 + v + v^2.$$

$$\frac{dv}{dx} = \frac{1+v^2}{x}.$$

$$\frac{1}{1+v^2}dv = \frac{1}{x}dx.$$

$$\int \frac{1}{1+v^2}dv = \int \frac{1}{x}dx.$$

$$\arctan v = \ln|x| + C.$$

$$v = \tan(\ln|x| + C).$$

$$y = vx = x\tan(\ln|x| + C).$$

(e) $y' + 2xy = \frac{6x}{y^2}$

solution)

$$y' + 2xy = \frac{6x}{y^2}\ :\ \text{Bernoulli equation with } n = -2.$$

Use a substitution $v = y^{1-n} = y^3$.

Then,

$$y = v^{\frac{1}{3}},\ \ \frac{dy}{dx} = \frac{1}{3}v^{-\frac{2}{3}}\frac{dv}{dx}.$$

$$y' + 2xy = \frac{6x}{y^2} \Rightarrow \frac{1}{3}v^{-\frac{2}{3}}\frac{dv}{dx} + 2xv^{\frac{1}{3}} = 6xv^{-\frac{2}{3}}.$$

$$\frac{dv}{dx} + 6xv = 18x\ :\ \text{linear first order equation}$$

Integrating factor $\rho(x) = e^{\int 6xdx} = e^{3x^2}$

$$\frac{d}{dx}\left(e^{3x^2}v\right) = 18xe^{3x^2}.$$

$$e^{3x^2}v = \int 18xe^{3x^2}dx$$

$$= 3e^{3x^2} + C\ \ (u = e^{3x^2},\ \ du = 6xe^{3x^2}).$$

$$v = 3 + Ce^{-3x^2}.$$

$$y = \left(3 + Ce^{-3x^2}\right)^{\frac{1}{3}}.$$

(f) $y' = (x + y - 7)^2$

solution)

Let $v = x + y - 7$. Then,

$$y = v - x + 7,\ \ \frac{dy}{dx} = \frac{dv}{dx} - 1.$$

$$y' = (x + y - 7)^2 \Rightarrow \frac{dv}{dx} - 1 = v^2.$$

$$\frac{dv}{dx} = v^2 + 1\ :\ \ \text{separable}$$

$$\frac{1}{v^2 + 1}dv = dx.$$

$$\int \frac{1}{v^2 + 1}dv = \int dx.$$

$$\arctan v = x + C.$$

$$v = \tan(x + C).$$

$$y = \tan(x + C) - x + 7$$

Problem 6 Solve the following initial value problems

(a) $xy' + 2y = 4x^2,\ \ y(1) = 2.$

solution)

$$
\begin{aligned}
xy' + 2y &= 4x^2 \ : \ \text{first linear equation} \\
y' + \frac{2}{x}y &= 4x. \\
\text{Integrating factor } \rho(x) &= e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2. \\
\frac{d}{dx}(x^2y) &= 4x^3. \\
x^2y &= \int 4x^3 dx = x^4 + C. \\
y &= x^2 + \frac{C}{x^2}. \\
y(1) &= 2;\ \ 1 + \frac{C}{1} = 2 \Rightarrow C = 1. \\
y &= x^2 + \frac{1}{x^2}.
\end{aligned}
$$

(b) $(6xy^3 + 2y^4) + (9x^2y^2 + 8xy^3)y' = 0,\ \ y(1) = 1.$

solution) Rewrite it as

$$
\begin{aligned}
(6xy^3 + 2y^4)dx + (9x^2y^2 + 8xy^3)dy &= 0. \\
\text{Let } M &= 6xy^3 + 2y^4 \text{ and } N = 9x^2y^2 + 8xy^3. \\
\text{Then, } M_y &= 18xy^2 + 8y^3 \text{ and } N_x = 18xy^2 + 8y^3. \Rightarrow M_y = N_x. \\
\text{Hence, there exists } F(x,y) \text{ such that } F_x &= 6xy^3 + 2y^4 \text{ and } F_y = 9x^2y^2 + 8xy^3. \\
F(x,y) &= \int (6xy^3 + 2y^4)dx + g(y) = 3x^2y^3 + 2xy^4 + g(y). \\
\Rightarrow \frac{\partial F}{\partial y} &= 9x^2y^2 + 8xy^3 + \frac{dg}{dy} = 9x^2y^2 + 8xy^3. \\
\frac{dg}{dy} &= 0 \Rightarrow g(y) = C. \\
F(x,y) &= 3x^2y^3 + 2xy^4 + C = D. \\
\text{Hence, a solution is } 3x^2y^3 + 2xy^4 &= C \text{ for some constant } C. \\
y(1) &= 1;\ \ 3 + 2 = C. \\
\text{Thus, solution is given by } 3x^2y^3 + 2xy^4 &= 5 \text{ implicitly.}
\end{aligned}
$$

Problem 7 Mixture Problem: Go over the homework problem 36a) in Section 1.5

Problem 8 Consider a body that moves horizontally through a medium whose resistance is proportional to the square of the velocity v, so that $\frac{dv}{dt} = -kv^2$. Assume that $v(0) = v_0$ and initial position $x(0) = x_0$.

a. Show that $v(t) = \frac{v_0}{1+v_0kt}$.

solution)

$$\frac{dv}{dt} = -kv^2 \ : \ \text{separable}$$

$$-\frac{1}{v^2}dv = kdt.$$

$$\int -\frac{1}{v^2}dv = \int kdt.$$

$$\frac{1}{v} = kt + C.$$

$$v = \frac{1}{kt+C}.$$

$$v(0) = v_0; \ \frac{1}{C} = v_0 \Rightarrow C = \frac{1}{v_0}.$$

$$\text{Hence, } v = \frac{1}{kt + \frac{1}{v_0}} = \frac{v_0}{v_0kt+1}.$$

b. Show that its position function $x(t) = x_0 + \frac{1}{k}\ln(1+v_0)$.

solution)

$$x(t) = \int v(t)dt = \int \frac{v_0}{1+v_0kt}dt$$

$$= \frac{1}{k}\ln|1+v_0kt| + C, \ \ (u = 1+v_0kt, \ du = v_0kdt).$$

$$x(0) = x_0; \ \frac{1}{k}\ln|1| + C = x_0 \Rightarrow C = x_0.$$

$$\text{Hence, } x(t) = x_0 + \frac{1}{k}\ln(1+v_0).$$

Problem 9 Consider a fish population $P(t)$ given by the following model $\frac{dP}{dt} = P(P-8)$, $P(0) = 5$

(a) Find the critical points.

solution)

$$P(P-8) = 0.$$
$$P = 0, 8.$$

(b) Use a phase diagram to determine each critical point is stable or unstable.

solution) It is similar to the homework problem 6 of section 2.1. Please refer to the hand drawn figure in the homework solutions.

$P = 0$: stable.

$P = 8$: unstable

(c) What can we expect as time goes?

solution) It will extinct as time goes. Again, please refer to the figure in the homework solution.

(d) Solve explicitly for $P(t)$.

solution)

$$\frac{dP}{dt} = P(P-8) : \text{ separable}$$
$$\frac{1}{P(P-8)} dP = dt.$$
$$\frac{1}{8}\int\left(\frac{1}{P-8} - \frac{1}{P}\right)dP = \int dt.$$
$$\ln\left|\frac{P-8}{P}\right| = 8t + C.$$
$$\left|\frac{P-8}{P}\right| = e^{8t+C}.$$
$$\frac{P-8}{P} = Ce^{8t}.$$
$$P - 8 = Ce^{8t}P \Rightarrow (1 - Ce^{8t})P = 8.$$
$$P = \frac{8}{1 - Ce^{8t}}.$$
$$P(0) = 5; \quad \frac{8}{1-C} = 5 \Rightarrow 8 = 5 - 5C \Rightarrow C = -\frac{3}{5}.$$
$$\text{Hence, } P(t) = \frac{8}{1 + \frac{3}{5}e^{8t}} = \frac{40}{5 + 3e^{8t}}.$$