Math 385 B1 Exam 2
Name (Please Print.): Key

Problem 1 (4 points each) Find a general solution of the following homogeneous differential equations.

(a) $y'' - y = 0$
solution)

$$r^2 - 1 = 0 \Rightarrow r = \pm 1.$$
$$y = c_1 e^x + c_2 e^{-x}.$$

(b) $y'' - 2y' + 3y = 0.$
solution)

$$r^2 - 2r + 3 = 0 \Rightarrow r = 1 \pm \sqrt{2}\, i.$$
$$y = e^x(c_1 \cos\sqrt{2}\, x + c_2 \sin\sqrt{2}\, x).$$

(c) $y'' - 6y' + 9y = 0.$
solution)

$$r^2 - 6r + 9 = 0 \Rightarrow r = 3, 3.$$
$$y = e^{3x}(c_1 + c_2 x).$$

Problem 2 (8 points) Find the appropriate form of a particular solution of $(D-2)^2(D^2 - 4D + 5)y = 7e^{2x}\cos x + xe^{2x} + 10.$ Do not evaluate the coefficients.
solution)

$$(r-2)^2(r^2 - 4r + 5) = 0 \Rightarrow r = 2, 2, 2 \pm i.$$
$$y_1 = e^{2x}, y_2 = xe^{2x}, y_3 = e^{2x}\cos x, y_4 = e^{2x}\sin x.$$

Determine the particular solution for each $7e^{2x}\cos x, xe^{2x}, 10.$

$$f_1(x) = 7e^{2x}\cos x \;:\; y_{p_1} = xe^{2x}(A\cos x + B\sin x).$$
$$f_2(x) = xe^{2x} \;:\; y_{p_2} = x^2(Cx + D)e^{2x}$$
$$f_3(x) = 10 \;:\; y_{p_3} = E.$$

Hence,

$$\begin{aligned} y_p &= y_{p_1} + y_{p_2} + y_{p_3} \\ &= xe^{2x}(A\cos x + B\sin x) + x^2(Cx + D)e^{2x} + E. \end{aligned}$$

Problem 3 (11 points) Find the general solution of $y'' - 2y' + y = \frac{e^x}{1+x^2}$ by using the method of variation of parameters.
solution)

$$r^2 - 2r + 1 = 0 \Rightarrow r = 1,1 \Rightarrow y_1 = e^x, y_2 = xe^x.$$

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}.$$

$$u_1 = -\int \frac{xe^x \cdot \frac{e^x}{1+x^2}}{e^{2x}} dx = -\int \frac{x}{1+x^2} dx = -\frac{1}{2}\ln(1+x^2),$$

$$u_2 = \int \frac{e^x \cdot \frac{e^x}{1+x^2}}{e^{2x}} dx = \int \frac{1}{1+x^2} dx = \arctan x.$$

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{2} e^x \ln(1+x^2) + xe^x \arctan x.$$

$$y = y_c + y_p = c_1 e^x + c_2 xe^x - \frac{1}{2} e^x \ln(1+x^2) + xe^x \arctan x.$$

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Problem 4 (9 points) Find the steady periodic solution of $2x'' + 3x' + 2x = 36\cos 2t$, and write the solution in $C\cos(\varpi t - \alpha)$ form with $C > 0$ and $0 < \alpha < 2\pi$.
solution)

$$\begin{aligned}
x_{\text{sp}}(t) &= A\cos 2t + B\sin 2t. \\
x'_{\text{sp}}(t) &= 2B\cos 2t - 2A\sin 2t. \\
x''_{\text{sp}}(t) &= -4A\cos 2t - 4B\sin 2t. \\
2x''_{\text{sp}} + 3x'_{\text{sp}} + 2x_{\text{sp}} &= (-6A + 6B)\cos 2t + (-6A - 6B)\sin 2t \\
&= 36\cos 2t. \\
-6A + 6B &= 36, -6A - 6B = 0 \Rightarrow A = -3, B = 3. \\
x_{\text{sp}}(t) &= -3\cos 2t + 3\sin 2t = \sqrt{9+9}\cos(2t - \frac{3\pi}{4}) \\
&= 3\sqrt{2}\cos(2t - \frac{3\pi}{4}).
\end{aligned}$$

Problem 5 Consider the following forced-undamped oscillation $x'' + 4x = \cos\omega t$.
(a) (10 points) Solve the given equation by considering two separate cases $\omega = 2$ and $\omega \neq 2$.
solution)

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow x_c = c_1\cos 2t + c_2\sin 2t.$$

$$\omega \neq 2 : \ x_p = A\cos\omega t \Rightarrow x''_p = -A\omega^2\cos\omega t.$$

$$x''_p + 4x_p = (-A\omega^2 + 4A)\cos\omega t = \cos\omega t.$$

$$(4-\omega^2)A = 1 \Rightarrow A = \frac{1}{4-\omega^2}.$$

$$x_p(t) = \frac{1}{4-\omega^2}.\cos\omega t.$$

Hence, $x(t) = x_c(t) + x_p(t) = c_1\cos 2t + c_2\sin 2t + \frac{1}{4-\omega^2}.\cos\omega t.$

$\omega = 2$: Due to a duplication in x_c, we need to set $x_p = t(A\cos 2t + B\sin 2t)$.

$$x''_p = 2(-2A\sin 2t + B\cos 2t) + t(-4A\cos 2t - 4B\sin 2t).$$

$$x''_p + 4x_p = -4A\sin 2t + 4B\cos 2t = \cos 2t.$$

$$-4A = 0, 4B = 1 \Rightarrow A = 0, B = \frac{1}{4}.$$

$$x_p(t) = \frac{1}{4}t\sin 2t.$$

Hence, $x(t) = x_c(t) + x_p(t) = c_1\cos 2t + c_2\sin 2t + \frac{1}{4}t\sin 2t.$

(b) (2 points) For which values of ω does the resonance occur?
solution) For the values of $\omega \neq 2$, the solution oscillates.
But if $\omega = 2$, the solution grows without a bound due to $\frac{1}{4}t\sin 2t$. Hence, the resonance occurs when $\omega = 2$.

Problem 6 (8 points) Find all the **positive** eigenvalues and the associated eigenfunctions of

$$y'' + \lambda y = 0;\ y(0) = 0,\ y'(1) = 0$$

solution)

$$\lambda > 0:\ r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}\, i.$$

$$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x.$$

The question is as follows: for which values of λ, there exists (c_1, c_2) such that $(c_1, c_2) \neq (0,0)$.

$$y' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x.$$

$$y(0) = 0:\ c_1 = 0.$$

$$y'(1) = c_2\sqrt{\lambda}\cos\sqrt{\lambda} = 0.$$

Since $c_1 = 0$, we do not want $c_2 = 0$. Hence, $\cos\sqrt{\lambda} = 0$.
Hence,

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2},\ n = 1,2,\ldots.$$

If $\sqrt{\lambda} = \frac{(2n-1)\pi}{2}$, the corresponding solution y is $y = c_2 \sin\frac{(2n-1)\pi}{2}x$.
Hence,

$$\text{eigenvalues are } \lambda_n = \frac{(2n-1)^2\pi^2}{4},\ n = 1,2,\ldots.$$

$$\text{eigenfunctions are } y_n = \sin\frac{(2n-1)\pi}{2}x,\ n = 1,2,\ldots$$