Practice Problems: Chapter 3

Problem 1 Find a general solution of the following homogeneous differential equations.

(a) $y'' - 7y' + 12y = 0$

solution)

$$\begin{aligned} r^2 - 7r + 12 &= 0. \\ (r-3)(r-4) &= 0. \\ r &= 3, 4 \\ y &= c_1 e^{3x} + c_2 e^{4x}. \end{aligned}$$

(b) $y^{(3)} - 2y^{(2)} + y^{(1)} = 0$

solution)

$$\begin{aligned} r^3 - 2r^2 + r &= 0. \\ r(r-1)^2 &= 0. \\ r &= 0, 1, 1 \\ y &= c_1 + c_2 e^x + c_3 x e^x. \end{aligned}$$

(c) $y'' - 2y' + 2y = 0$

solution)

$$\begin{aligned} r^2 - 2r + 2 &= 0. \\ r &= 1 \pm i \\ y &= c_1 e^x \cos x + c_2 e^x \sin x = e^x (c_1 \cos x + c_2 \sin x). \end{aligned}$$

(d) $(D-3)^2(D^2 - 6D + 13)y = 0,$ where $D = \dfrac{\partial}{\partial x}$.

solution)

$$\begin{aligned} (r-3)^2(r^2 - 6r + 13) &= 0. \\ r &= 3, 3, 3 \pm 2i \\ y &= c_1 e^{3x} + c_2 x e^{3x} + e^{3x}(c_3 \cos 2x + c_4 \sin 2x). \end{aligned}$$

Problem 2 Find the appropriate form of a particular solution of the following nonhomogeneous differential equations. Do not evaluate the coefficients.

(a) $y'' - 2y' - 3y = 6$

solution)

$r^2 - 2r - 3 = (r-3)(r+1) = 0,\ r = 3, -1 \Rightarrow$ two linearly independent solutions in y_c are e^{3x}, e^{-x}.

$$y_p = A.$$

(b) $y^{(5)} - y^{(3)} = 6$

solution)

$$r^5 - r^3 = r^3(r-1)(r+1) = 0,\ r = 0,0,0,1,-1$$

$\Rightarrow$ five linearly independent solutions in y_c are $1, x, x^2, e^x, e^{-x}$.

Since $f(x) = 6,$ we may try $y_p = A.$ But to eliminate duplication in $y_{c,}$ we need to multiply by x^3.

$$y_p = Ax^3.$$

(c) $y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$

solution) Set up a particular solution for each e^x and $2x^2 - 5.$ We may try with $y_p = Ae^x + (Bx^2 + Cx + D).$

But because of duplication in y_c,

$$y_p = Axe^x + x^3(Bx^2 + Cx + D).$$

(d) $y'' - 2y' + 2y = e^x \sin x$

solution)

Two indendent solutions in y_c are $y_1 = e^x \cos x,\ y_2 = e^x \sin x$ (see problem 1c).

Since $f(x) = e^x \sin x,$ we want to try $e^x(A\cos x + B\sin x).$ But again due to duplication in $y_{c,}$

$$y_p = xe^x(A\cos x + B\sin x).$$

(e) $y'' - 7y' + 12y = (x+2)e^{3x}$

solution)

Two indendent solutions in y_c are $y_1 = e^{3x},\ y_2 = e^{4x}$ (see problem 1a).

Since $f(x) = (x+2)e^{3x},$ we want to try $(Ax + B)e^{3x}.$ But again due to duplication in $y_{c,}$

$$y_p = x(Ax+B)e^{3x} = (Ax^2 + Bx)e^{3x}.$$

(f) $(D-3)^2(D^2 - 6D + 13)y = 5e^{3x}\cos 2x + x^2e^{3x}$

solution)

Indendent solutions in y_c are $y_1 = e^{3x},\ y_2 = xe^{3x}\ , y_3 = e^{3x}\cos 2x, y_4 = e^{3x}\sin 2x$ (see problem 1d).

Since $f(x) = 5e^{3x}\cos 2x + x^2e^{3x},$ we want to try $e^{3x}(A\cos 2x + B\sin 2x) + e^{3x}(Cx^2 + Dx + E).$

But again due to duplication in $y_{c,}$

$$y_p = xe^{3x}(A\cos 2x + B\sin 2x) + x^2e^{3x}(Cx^2 + Dx + E).$$

Problem 3 Find a particular solution of $y'' + 9y = \sec 3x$.
solution)

$$r^2 + 9 = 0.$$
$$r = \pm 3i.$$
$$y_1 = \cos 3x,\ y_2 = \sin 3x.$$

$$y_p = u_1 y_1 + u_2 y_2,$$

where $u_1 = -\int \frac{y_2 \cdot f}{W} dx, u_2 = \int \frac{y_1 \cdot f}{W} dx,$ and $f(x,y) = x^{-2}e^{-2x}.$

$$W(y_1, y_2) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3.$$

$$u_1 = -\int \frac{\sec 3x \sin 3x}{3} dx = -\frac{1}{3}\int \frac{\sin 3x}{\cos 3x} dx = \frac{1}{9}\ln|\cos 3x|,\ (u = \cos 3x, du = -3\sin 3x dx)$$
$$u_2 = \int \frac{\sec 3x \cos 3x}{3} dx = \int \frac{1}{3} dx = \frac{1}{3}x.$$

Hence,

$$y_p = \frac{1}{9}\cos 3x \cdot \ln|\cos 3x| + \frac{1}{3}x\sin 3x$$
$$(= -\frac{1}{9}\cos 3x \cdot \ln|\sec 3x| + \frac{1}{3}x\sin 3x).$$

Problem 4 (a) Consider $x^2y'' + xy' - y = 0,\ x > 0.$ Assuming that the solution is in $y = x^r$ form, find r.
solution)

$$y = x^r, y' = rx^{r-1}, y'' = r(r-1)x^{r-2}.$$

$$\begin{aligned} x^2y'' + xy' - y &= 0 \\ r(r-1)x^r + rx^r - x^r &= x^r(r(r-1) + r - 1) = 0. \\ r(r-1) + r - 1 &= 0. \\ (r-1)(r+1) &= 0. \\ r &= 1, -1. \end{aligned}$$

Hence, solutions are $y = x$ and $y = x^{-1}$.

(b) Find a general solution of $x^2y'' + xy' - y = 9x^{-10}$.
solution) Rewrite the equation as

$$y'' + x^{-1}y' - x^{-2}y = 9x^{-12}.$$

$$y_1 = x, y_2 = x^{-1}.$$

$$y_p = u_1y_1 + u_2y_2,$$

where $u_1 = -\int \frac{y_2 \cdot f}{W}dx, u_2 = \int \frac{y_1 \cdot f}{W}dx,$ and $f(x,y) = x^{-2}e^{-2x}$.

$$W(y_1, y_2) = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1}.$$

$$u_1 = -\int \frac{x^{-1}9x^{-12}}{-2x^{-1}}dx = \frac{9}{2}\int x^{-12}dx = -\frac{9}{22}x^{-11}.$$

$$u_2 = \int \frac{x9x^{-12}}{-2x^{-1}}dx = -\frac{9}{2}\int x^{-10}dx = \frac{1}{2}x^{-9}.$$

So,

$$y_p = -\frac{9}{22}x^{-11}x + \frac{1}{2}x^{-9}x^{-1} = \frac{1}{11}x^{-10}.$$

Hence,

$$y = y_c + y_p = c_1x + c_2x^{-1} + \frac{1}{11}x^{-10}.$$

Problem 5 Consider the free damped oscillation $x'' + 2x' + 5x = 0$ with initial position $x(0) = 1$ and initial velocity $x'(0) = -1 - 2\sqrt{3}$.

(a) Find its position function.

solution)

$$\begin{aligned} r^2 + 2r + 5 &= 0. \\ r &= -1 \pm 2i. \\ x(t) &= e^{-t}(c_1 \cos 2t + c_2 \sin 2t). \\ x'(t) &= -e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + e^{-t}(2c_2 \cos 2t - 2c_1 \sin 2t). \\ x(0) &= 1;\ c_1 = 1. \\ x'(0) &= -c_1 + 2c_2 = -1 - 2\sqrt{3} \Rightarrow c_2 = -\sqrt{3}. \\ x(t) &= e^{-t}(\cos 2t - \sqrt{3} \sin 2t). \end{aligned}$$

(b) Find the time lag, envelope curves.

solution) Write $x(t)$ in $Ce^{-t}\cos(\omega t - \alpha)$. Then, we get

$$\begin{aligned} x(t) &= e^{-t}(\cos 2t - \sqrt{3} \sin 2t) \\ &= \sqrt{1+3}\, e^{-t} \cos(2t - \frac{5\pi}{3}) \\ &= 2e^{-t} \cos 2(t - \frac{5\pi}{6}). \end{aligned}$$

Note that the angle α is on the Quad IV, and $\arctan(\frac{\sqrt{3}}{1}) = \frac{\pi}{3}$.

Hence,

Time lag is $\frac{5\pi}{6}$, and the envelope curves are $x = \pm 2e^{-t}$.

(c) Determine the behavior of $x(t)$ as $t \to \infty$.

solution) It decays while oscillating. Note that $\lim_{t\to\infty} 2e^{-t} \cos 2(t - \frac{5\pi}{6}) = 0$.

Problem 6 Consider following forced-undamped oscillation $x'' + 9x = \cos\omega t$.
(a) Solve the given differential equation by considering two separate cases $\omega = 3$ and $\omega \neq 3$.
solution) $r^2 + 9 = 0 \Rightarrow r = \pm 3i$.

$$x_c = c_1 \cos 3t + c_2 \sin 3t.$$

i) $\omega \neq 3$: $x_p = A\cos\omega t + B\sin\omega t$.
Since there is no x' in the differential equation, it is easy to see that $B = 0$.

$$\begin{aligned} x_p &= A\cos\omega t. \\ x_p'' &= -A\omega^2 \cos\omega t. \\ x_p'' + 9x_p &= (-A\omega^2 + 9A)\cos\omega t = \cos\omega t. \\ A(-\omega^2 + 9) &= 1. \\ A &= \frac{1}{9-\omega^2}. \end{aligned}$$

Hence,

$$x(t) = c_1 \cos 3t + c_2 \sin 3t + \frac{1}{9-\omega^2}\cos\omega t.$$

ii) $\omega = 3$: To eliminate duplication in x_c, we need to try

$$\begin{aligned} x_p &= t(A\cos 3t + B\sin 3t). \\ x_p'' &= 2(-3A\sin 3t + 3B\cos 3t) + t(-9A\cos 3t - 9B\sin 3t). \\ x_p'' + 9x_p &= -6A\sin 3t + 6B\cos 3t = \cos 3t. \\ -6A &= 0, 6B = 1. \\ A &= 0, B = \frac{1}{6}. \end{aligned}$$

Hence,

$$x(t) = c_1 \cos 3t + c_2 \sin 3t + \frac{1}{6} t \sin 3t.$$

(b) For which value(s) of ω does the resonance occur, i.e., the solution grows and oscillates without bound?
solution) The resonance occurs when $\omega = 3$, since $\frac{1}{6} t \sin 3t$ grows without bound.

Problem 7 Consider the damped forced oscillation $x'' + 6x' + 9x = 13\cos 2t$.
(a) Find the transient solution $x_{\text{tr}}(t)$, and determine the behavior as $t \to \infty$.
solution)

$$r^2 + 6r + 9 = (r+3)^2 = 0.$$
$$r = -3, -3.$$
$$x_{\text{tr}}(t) = c_1 e^{-3t} + c_2 t e^{-3t} = e^{-3t}(c_1 + c_2 t).$$

As $t \to \infty$, $x_{\text{tr}}(t) \to 0$.

(b) Find the steady periodic solution $x_{\text{sp}}(t)$.
solution)

$$x_{\text{sp}}(t) = A\cos 2t + B\sin 2t.$$
$$x'_{\text{sp}} = 2B\cos 2t - 2A\sin 2t.$$
$$x''_{\text{sp}} = -4A\cos 2t - 4B\sin 2t.$$
$$x''_{\text{sp}} + 6x'_{\text{sp}} + 9x_{\text{sp}} = (-4A + 12B + 9A)\cos 2t + (-4B - 12A + 9B)\sin 2t = 18\cos 2t.$$
$$5A + 12B = 13,$$
$$-12A + 5B = 0.$$
$$A = \frac{5}{13},\ B = \frac{12}{13}.$$
$$x_{\text{sp}}(t) = \frac{5}{13}\cos 2t + \frac{12}{13}\sin 2t.$$

Problem 8 Find all the non-negative eigenvalues and the associated eigenfunctions of

$$y'' + \lambda y = 0;\ y'(0) = 0,\ y'(\pi) = 0.$$

solution)

i) $\lambda = 0;\ r^2 = 0, r = 0,0 \Rightarrow y = A + Bx.$

$y'(0) = 0,\ y'(\pi) = 0;\ B = 0.$

Hence, $y = A$ is an eignefunction associated with $\lambda = 0.$

Note that we typically choose a constant $A = 1,$ and say that $y = 1$ is the eigenfunction associated with $\lambda = 0.$

ii) $\lambda > 0;\ r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\, i.$

$$y = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x).$$

$$y' = B\sqrt{\lambda}\,\cos(\sqrt{\lambda}\,x) - A\sqrt{\lambda}\,\sin(\sqrt{\lambda}\,x).$$

$$y'(0) = B\sqrt{\lambda} = 0 \Rightarrow B = 0.$$

$$y'(\pi) = -A\sqrt{\lambda}\,\sin(\sqrt{\lambda}\,\pi) = 0 \Rightarrow \sin(\sqrt{\lambda}\,\pi) = 0 \text{ if } A \neq 0.$$

Hence, $\sqrt{\lambda} = n\ (n = 1, 2, \ldots).$

Thus, eigenvalues are $\lambda = n^2\ (n = 1, 2, \ldots)$ and its associated eigenfunctions are $y = \cos nx$ (by simply taking $A = 1$)