Practice Problems: Chapter 9 except 9.7

Problem 1 Let $f(t)$ be the periodic function with period 4 defined by

$$f(t) = \begin{cases} 0 & \text{if } -2 < x < 0 \\ 2 & \text{if } 0 < x < 2. \end{cases}$$

Find its Fourier series.

solution) $L = 2$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_0^2 2dx = 2.$$

$$a_n = \frac{1}{2}\int_0^2 2\cos\frac{n\pi x}{2}dx = 0,\ n = 1,2,\ldots$$

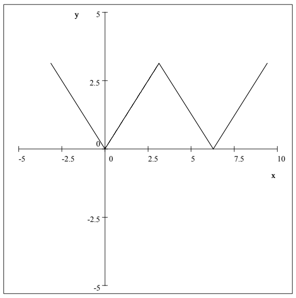
$$b_n = \frac{1}{2}\int_0^2 2\sin\frac{n\pi x}{2}dx = \frac{-2}{n\pi}\cos\frac{n\pi x}{2}\Big|_{x=0}^{2}$$

$$= \frac{2}{n\pi}(1-\cos n\pi) = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & n \text{ odd} \end{cases}$$

Hence,

$$f(t) = 1 + \sum_{n \text{ odd}} \frac{4}{n\pi}\sin\frac{n\pi x}{2}.$$

Problem 2 Let $f(t) = t$ for $0 < t < \pi$. Sketch the even extension of $f(t)$, and find the Fourier cosine series of $f(t)$.
solution)



$L = \pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t dt = \frac{2}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi} = \pi$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} t\cos nt dt \\ &= \frac{2}{\pi}\left(\left[\frac{1}{n}t\sin nt\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nt dt\right) \\ &= \frac{2}{\pi n^2}\cos nt|_0^{\pi} = \frac{2}{\pi n^2}((-1)^n - 1) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Hence,

$$f(t) = \frac{\pi}{2} + \sum_{n \text{ odd}} \frac{-4}{\pi n^2}\cos nt.$$

Problem 3 Let $f(t)$ be the function of period 2π defined by

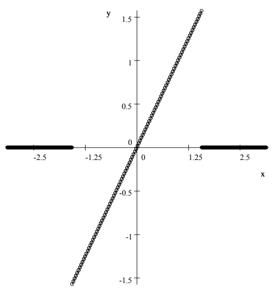
$$f(t) = \begin{cases} 0, & -\pi < t < -\frac{\pi}{2} \\ t, & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < t < \pi. \end{cases}$$

We know that its Fourier series is

$$\sum_{n \text{ odd}} \frac{2}{n^2\pi} \sin\left(\frac{n\pi}{2}\right) \sin nt + \sum_{n \text{ even}} -\frac{1}{n} \cos\left(\frac{n\pi}{2}\right) \sin nt.$$

a) Sketch the graph. To what value does the series converge at $t = \frac{\pi}{2}$

solution)



Note that at the point of discoutinuity, the Fourier series converges to $\frac{f(t+) \ + f(t-)}{2}$.

Since $f(t)$ is not continuous at $t = \frac{\pi}{2}$, the Fourier series converges to

$$\frac{f(\frac{\pi}{2}+) \ + f(\frac{\pi}{2}-)}{2} = \frac{\frac{\pi}{2} + 0}{2} = \frac{\pi}{4}.$$

b) Show that $\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$.

solution) Plug in $t = \frac{\pi}{2}$. Then,

$$\frac{\pi}{4} = \sum_{n \text{ odd}} \frac{2}{\pi n^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) + \sum_{n \text{ even}} \frac{-1}{n} \cos\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right)$$

Since $\sin(\frac{n\pi}{2}) = 0$ for n even and ± 1 for n odd, it follows that

$$\frac{\pi}{4} = \sum_{n \text{ odd}} \frac{2}{\pi n^2} sin^2\left(\frac{n\pi}{2}\right) = \sum_{n \text{ odd}} \frac{2}{\pi n^2}$$

Multiplying through by $\frac{1}{2}$, we have

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Problem 4 Find the formal Fourier series solution of

$$y'' + 4y = \sum_{n \text{ odd}} \frac{40}{n\pi} \sin nt$$

solution)

$$y = y_c + y_p.$$

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow y_c = c_1 \cos 2t + c_2 \sin 2t.$$

To find a particular solution, we find a solution for each $\frac{40}{n\pi} \sin nt$ and use the method of superposition. Since there is no y' term in the equation and $\sin nt$ (n odd) does not duplicate with y_c, we try with

$$y_p(t) = \sum_{n \text{ odd}} B_n \sin nt$$

Then,

$$y_p'' = \sum_{n \text{ odd}} -n^2 B_n \sin nt.$$

Plug in y_p, y_p'', we get

$$\sum_{n \text{ odd}} (4 - n^2) B_n \sin nt = \sum_{n \text{ odd}} \frac{40}{n\pi} \sin nt$$

Hence,

$$(4 - n^2) B_n = \frac{40}{n\pi}, \ n \text{ odd}$$

$$n = \frac{40}{n\pi(4 - n^2)}, \ n \text{ odd}.$$

Thus,

$$\boxed{y(t) = y_c(t) + y_p(t) = c_1 \cos 2t + c_2 \sin 2t + \sum_{n \text{ odd}} \frac{40}{n\pi(4-n^2)} \sin nt.}$$

Problem 5 Consider the following boundary value problem

$$u_t = 2u_{xx}, \quad 0 < x < 2,$$

$$u_x(0,t) = u_x(2,t) = 0, \text{ (endpoints conditions)}$$

$$u(x,0) = x. \text{(initial condition)}$$

(a) Assume that $u(x,t) = X(x)T(t)$. Rewriing the differential equation and the endpoint conditions in terms of $X(x)$ and $T(t)$, find separate equations and endpoints conditions (if any) for $X(t)$ and $T(t)$.

solution)

$$u(x,t) = X(x)T(t)$$

$$XT' = 2X''T$$

$$\frac{X''}{X} = \frac{T'}{2T} = -\lambda.$$

$$X'' + \lambda X = 0, \ \frac{T'}{2T} = -\lambda.$$

Now the endpoints conditions will be converted to

$$X'(0)T(t) = 0 \Rightarrow X'(0) = 0$$

$$X'(2)T(t) = 0 \Rightarrow X'(2) = 0$$

Hence, we have

$X'' + \lambda X = 0$
$X'(0) = 0, \ X'(2) = 0.$

and

$\frac{T'}{2T} = -\lambda.$

(b) Find the solution satisfying endpoints conditions assuming that eigenvalues are non negative.

solution) First solve the eigenvalue problem

$X'' + \lambda X = 0$
$X'(0) = 0, \ X'(2) = 0.$

Characteristic equation is $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$.

Note that we have three different types of the solutions depending on the signs of λ :

i) $\lambda = 0 : X(x) = c_1 + c_2 x \Rightarrow X'(x) = c_2$.

Since $X'(0) = 0, \ X'(2) = 0, \ c_2 = 0$.

Hence, $X(x) = c_1$.

Thus, we have

eigenvalue: $\lambda_0 = 0$

eigenfunction: $X_0 = 1$.

ii) $\lambda > 0 : X(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x \Rightarrow X'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x.$

$$X'(0) = 0,\ X'(2) = 0 \Rightarrow c_2\sqrt{\lambda} = 0, \quad -c_1\sqrt{\lambda}\sin 2\sqrt{\lambda} + c_2\sqrt{\lambda}\cos 2\sqrt{\lambda} = 0$$
$$\Rightarrow c_2 = 0,\ -c_1\sqrt{\lambda}\sin 2\sqrt{\lambda} = 0.$$

Since we do not want c_1 to be 0, $\sin 2\sqrt{\lambda} = 0$.
Hence,

$$2\sqrt{\lambda} = n\pi,\ n = 1,2,\ldots.$$
$$\Rightarrow \lambda = \frac{n^2\pi^2}{4},\ n = 1,2,\ldots.$$

Hence, $X(x) = c_1\cos\frac{n\pi x}{2}$.

Thus,

eigenvalues: $\lambda_n = \frac{n^2\pi^2}{4},\ n = 1,2,\ldots.$

eigenfunctions: $X_n(x) = \cos\frac{n\pi x}{2},\ n = 1,2,\ldots$

Now solve for T for each eigenvalue:

i) $\lambda_0 = 0$: $\frac{T'}{2T} = 0 \Rightarrow \frac{dT}{2T} = 0dt \Rightarrow \frac{1}{2}\ln|T| = C \Rightarrow T = e^{2C}$

Hence, $\boxed{T_0 = 1}$

ii) $\lambda_n = \frac{n^2\pi^2}{4},\ n = 1,2,\ldots.$:

$\frac{T'}{2T} = -\frac{n^2\pi^2}{4} \Rightarrow \frac{dT}{2T} = -\frac{n^2\pi^2}{4}dt \Rightarrow \ln|T| = -\frac{n^2\pi^2}{2}t + C \Rightarrow T = e^Ce^{-\frac{n^2\pi^2}{2}t}$

Hence, $\boxed{T_n = e^{-\frac{n^2\pi^2}{2}t}}$

Thus, the solutions are

$$u_0(x,t) = X_0(x)T_0(t) = 1,$$
$$u_n(x,t) = X_n(x)T_n(t) = \cos\frac{n\pi x}{2}e^{-\frac{n^2\pi^2}{2}t},\ n = 1,2,\ldots$$

By taking linear combination of thses solution, we find a general solution satisfying the endpoints conditions:

$$\boxed{u(x,t) = A_0 \cdot 1 + \sum_{n=1}^{\infty} A_n u_n(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\tfrac{n\pi x}{2}e^{-\frac{n^2\pi^2}{2}t}.}$$

(c) Find the solution satisfying both endpoints conditions and the initial condition.

solution)

$$u(x,t) = A_0 \cdot 1 + \sum_{n=1}^{\infty} A_n u_n(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{2}e^{-\frac{n^2\pi^2}{2}t}.$$

Since $u(x,0) = x$, we have

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{2} = x.$$

Expand x as the Fourier consien series $(L = 2)$ and compare coefficients, we get

$$A_0 = \frac{a_0}{2} = \frac{1}{2}\int_0^2 x dx = 1$$

$$A_n = a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2} dx = \frac{2}{n\pi}\left| x\sin\frac{n\pi x}{2}\right|_{x=0}^{2} - \frac{2}{n\pi}\int_0^2 \sin\frac{n\pi x}{2} dx$$

$$= \frac{4}{n^2\pi^2}\left|\cos\frac{n\pi x}{2}\right|_{x=0}^{2}$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{8}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases} .$$

Hence, the solution is

$$\boxed{u(x,t) = 1 - \sum_{n \text{ odd}} \frac{8}{\pi^2 n^2}\cos\frac{n\pi x}{2} e^{-\frac{n^2\pi^2}{2}t}.}$$

Problem 6 Consider the following boundary value problem

$$u_{tt} = u_{xx}, \quad 0 < x < 1,\ t > 0,$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

(a) Show that the solution $u(x,t)$ can be written as $u(x,t) = v(x,t) + w(x,t)$, where $v(x,t)$ is the solution of the same problem with $g(x) = 0$, and $w(x,t)$ is the solution of the same problem with $f(x) = 0$.

solution)

$$u_{tt} = v_{tt} + w_t = v_{xx} + w_{xx} = u_{xx},$$
$$u(0,t) = v(0,t) + w(0,t) = 0 + 0 = 0,\ \ u(1,t) = v(1,t) + w(1,t) = 0 + 0 = 0,$$
$$u(x,0) = v(x,0) + w(x,0) = f(x) + 0 = f(x),$$
$$u_t(x,0) = v_t(x,0) + w_t(x,0) = 0 + g(x) = g(x).$$

(b) Solve the above boundary value problem with $f(x) = 0$, and $g(x) = 4\sin 2\pi x + 4\sin 4\pi x$, assuming that the eigenvalus and eigenfunctions for $Y'' + \lambda Y = 0,\ Y(0) = 0,\ Y(L) = 0$ are $\lambda_n = \frac{n^2\pi^2}{L^2}$ and $Y_n = \sin\frac{n\pi y}{L},\ n = 1,2,\ldots$

solution) Set $u(x,t) = X(x)T(t)$.

Then,

$$XT'' = X''T \Rightarrow \frac{X''}{X} = \frac{T''}{T} = -\lambda.$$
$$X(0) = 0,\ X(1) = 0,\ T(0) = 0$$

So, we have two equations, one for X and the other for T.

$$\boxed{\begin{array}{l} X'' + \lambda X = 0 \\ X(0) = 0,\ X(1) = 0 \end{array}} \quad \text{and} \quad \begin{array}{l} T'' + \lambda T = 0. \\ T(0) = 0 \end{array}$$

There are nontrivial solutions

$$\boxed{X_n(x) = \sin\tfrac{n\pi x}{1} = \sin n\pi x \text{ when } \lambda_n = \frac{n^2\pi^2}{L^2} = n^2\pi^2 \text{ for } n = 1,2,\ldots.}$$

For such $\lambda_n = \frac{n^2\pi^2}{L^2} = n^2\pi^2$, we have

$$T'' + n^2\pi^2 T = 0,\ T(0) = 0$$
$$r^2 + n^2\pi^2 = 0 \Rightarrow r = n\pi i$$
$$T(t) = c_1\cos n\pi t + c_2\sin n\pi t.$$
$$T(0) = 0 :\ c_1 = 0.$$
$$T(t) = c_2\sin n\pi t.$$

Take $\boxed{T_n(t) = \sin n\pi t.}$

Hence, the fundamental set of solutions are $u_n(x,t) = X_n(x)T_n(t) = \sin n\pi x \sin n\pi t.$

Thus,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin n\pi x \sin n\pi t.$$

$$u_t(x,0) = 8\sin(\pi x)\cos(3\pi x) : \sum_{n=1}^{\infty} n\pi c_n \ \sin n\pi x \cos 0 = 4\sin 2\pi x + 4\sin 4\pi x.$$

$$\Rightarrow n\pi c_n = 4$$

Comparing similar terms, we deduce that

$2\pi c_2 = 4, \quad c_2 = \frac{2}{\pi}$
$4\pi c_4 = 4, \quad c_4 = \frac{1}{\pi}$
$c_n = 0$, otherwise.

Thus, the solution is

$u(x,t) = \frac{2}{\pi}\sin 2\pi x \sin 2\pi t + \frac{1}{\pi}\sin 4\pi x \sin 4\pi t.$