Practice Problems: Chapter 9 except 9.7

Problem 1 Let $f(t)$ be the periodic function with period 4 defined by

$$f(t) = \begin{cases} 0 & \text{if } -2 < x < 0 \\ 2 & \text{if } 0 < x < 2. \end{cases}$$

Find its Fourier series.

Problem 2 Let $f(t) = t$ for $0 < t < \pi$. Sketch the even extension of $f(t)$, and find the Fourier cosine series of $f(t)$.

Problem 3 Let $f(t)$ be the function of period 2π defined by

$$f(t) = \begin{cases} 0, & -\pi < t < -\frac{\pi}{2} \\ t, & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < t < \pi. \end{cases}$$

We know that its Fourier series is

$$\sum_{n \text{ odd}} \frac{2}{n^2\pi} \sin\left(\frac{n\pi}{2}\right) \sin nt + \sum_{n \text{ even}} -\frac{1}{n} \cos\left(\frac{n\pi}{2}\right) \sin nt.$$

a) Sketch the graph. To what value does the series converge at $t = \frac{\pi}{2}$?

b) Show that $\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$.

Problem 4 Find the formal Fourier series solution of

$$y'' + 4y = \sum_{n \text{ odd}} \frac{40}{n\pi} \sin nt.$$

Problem 5 Consider the following boundary value problem

$$u_t = 2u_{xx}, \quad 0 < x < 2,$$

$$u_x(0,t) = u_x(2,t) = 0, \text{ (endpoints conditions)}$$

$$u(x,0) = x. \text{ (initial condition)}$$

(a) Assume that $u(x,t) = X(x)T(t).$ Rewriing the differential equation and the endpoint conditions in terms of $X(x)$ and $T(t),$ find separate equations and endpoints conditions (if any) for $X(t)$ and $T(t).$

(b) Find the solution satisfying endpoints conditions assuming that eigenvalues are non negative.

(c) Find the solution satisfying both endpoints conditions and the initial condition.

Problem 6 Consider the following boundary value problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x). \end{aligned}$$

(a) Show that the solution $u(x,t)$ can be written as $u(x,t) = v(x,t) + w(x,t)$, where $v(x,t)$ is the solution of the same problem with $g(x) = 0$, and $w(x,t)$ is the solution of the same problem with $f(x) = 0$.

(b) Solve the above boundary value problem with $f(x) = 0$, and $g(x) = 4\sin 2\pi x + 4\sin 4\pi x$.