Practice Problems For the Final:

Problem 1 Find the general solutions of the following DEs.

a) $x^2y' - xy - y^2 = 0$

b) $(x^2 - 1)y' + (x - 1)y = 1$

c) $y^2y' + 2xy^3 = 6x$

d) $y' = \sqrt{x+y}$

Problem 2 A hemispherical bowl (with top radius 4) shaped water tank is slowly losing water at its lower end. As a result, the height of water in the tank, given by $y(t)$ satisfies

$$\frac{dy}{dt} = -\frac{1}{72}\frac{\sqrt{y}}{(8y - y^2)}.$$

a) Solve the DE for $y(t)$ when the tank is initially full.

b) How long does it take for the tank to be empty?

Problem 3 Determine whether the given functions are linearly independent or not.

a) $f(x) = e^x \sin x,\ g(x) = e^x \cos x.$

b) $f(x) = \sin 2x,\ g(x) = \sin x \cos x$ and $h(x) = e^x.$

Problem 4 Find the unique solution to the initial value problem

$$y'' - 7y' + 12y = (x+2)e^{3x}, \quad y(0) = 0, y'(0) = 2.$$

Problem 5 Find the general solution of the differential equation

$y''' - y'' + 4y' + 6y = e^{2x} + 2x.$

Problem 6 Find the general solution to the differential equation $y'' - 2y' + y = \frac{e^x}{1+x^2}$.

Problem 7 Find the Fourier series solution of the end point problem

$$x'' + 2x = 1$$

$$x(0) = 0, \quad x(1) = 0.$$

Problem 8 Find a particular solution of the following equations.

a) $x'' + 2x = \sin t$

b) $x'' + 2x = \sum_{n \text{ odd}} \frac{4}{n} \sin nt$. (Find a formal Fourier series solution.)

Problem 9 Consider the following eigenvalue problem

$$X'' + \lambda X = 0, \quad 0 < x < \pi$$

$$X(0) = 0, \quad X'(\pi) = 0.$$

Show that the eigenvalues λ_n and eigenfunctions X_n are given by

$$\lambda_n = \frac{(2n-1)^2}{2^2}, \quad X_n = \sin\frac{(2n-1)x}{2}, n = 1, 2, \ldots.$$

You may use the following fact: $\cos x = 0$ if and only if $x = \frac{n\pi}{2}, n = 1, 3, 5, \ldots$ (i.e, n: odd).

Problem 10 By separating the variables, solve the following wave equation

$$4u_{xx} = u_{tt}, \quad 0 < x < \pi, t > 0$$

$$u(0,t) = 0,\ u(\pi,t) = 0, t \geq 0.$$

$$u(x,0) = 1,\ u_t(x,0) = 0, \quad 0 < x < \pi.$$

Problem 11 Find the solution of the following problem. You are not required to determine the coefficients.

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 2, 0 < y < 2$$

$$u(x,0) = u(x,2) = 0$$

$$u(0,y) = 0.$$