Math 385 B1 Quiz 7

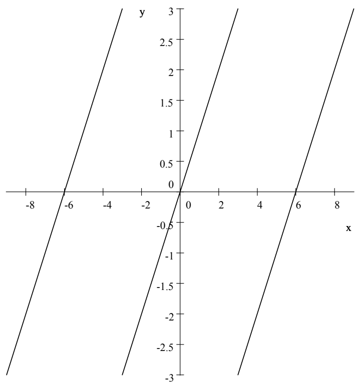
Name (Please Print.): Key

Problem 1 (9 points) Let $f(t)$ be a periodic function with period 6, and is given by

$$f(t) = t, \quad -3 < t < 3.$$

Find its Fouries seried, and write out 4 terms of its Fourier series.

solution) Note that $f(t)$ is odd, and hence $a_n = 0, \ n = 0, 1, 2, \ldots.$



$$L = 3$$

$$b_n = \frac{1}{3}\int_{-3}^{3} t\sin\frac{n\pi t}{3}dt = \frac{2}{3}\int_{0}^{3} t\sin\frac{n\pi t}{3}dt$$

$$= \frac{2}{3}\left[-\frac{3t}{n\pi}\cos\frac{n\pi t}{3}\bigg|_{t=0}^{3} + \frac{3}{n\pi}\int_0^3\cos\frac{n\pi t}{3}dt\right] = \frac{2}{3}\left[-\frac{9}{n\pi}\cos n\pi + \frac{9}{n^2\pi^2}\sin\frac{n\pi t}{3}\bigg|_{t=0}^{3}\right]$$

$$= -\frac{6}{n\pi}\cos n\pi = -\frac{6}{n\pi}(-1)^n.$$

Hence,

$$f(t) \sim \sum_{n=1}^{\infty} -\frac{6}{n\pi}(-1)^n \sin\frac{n\pi t}{3} = \frac{6}{\pi}\sin\frac{\pi}{3}t - \frac{1}{\pi}\sin\frac{2\pi}{3}t + \frac{2}{\pi}\sin\pi t - \frac{3}{2\pi}\sin\frac{4\pi}{3} + \cdots.$$

Problem 2 (4 points) Let $f(t)$ be a function of period 2π, and is given by

$$f(t) = t^2, \quad 0 < t < 2\pi.$$

Find its Fourier series. **Do Not evaluate the integrals**. Leave them as they are.
solution) $L = \pi$

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt,$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2 dt,$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt \, dt,$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt \, dt.$$

Problem 3 (2 points) Let $f(t)$ be a function of period 2 with $f(t) = t^2$ if $0 < t < 2$ and its Fourier series is given by

$$\frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin n\pi t}{n}.$$

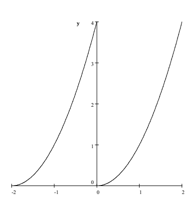
Using the convergence theorem of Fouries series, find the exact value of the following convergent series

$$\frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(hint: plug in $t = 0$, and use the theorem).

solution) Note that the Fourier series converges to $\frac{f(t+) + f(t-)}{2}$ at the point of discontinuity.
Since $f(t)$ is not continuous at $t = 0$, we get

$$\frac{f(0+) + f(0-)}{2} = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos n\pi 0}{n^2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin n\pi 0}{n} \Rightarrow$$
$$\frac{0+4}{2} = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$



Hence, $\frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} = 2.$