Math 385 B1 Quiz 8

Name (Please Print.): Key

Problem 1 (15 points) Find the steady periodic solution (particular solution) of

$$x'' + 5x = F(t),$$

where $F(t)$ is even function of period 4 such that $F(t) = \begin{cases} 3 & \text{if } 0 < t < 1 \\ -3 & \text{if } 1 < t < 2. \end{cases}$

solution) The Fourier series of $F(t)$ is in this form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2}$, since $F(t)$ is even and $L = 2$.

$$a_0 = \frac{2}{2}\left(\int_0^1 3dt + \int_1^2 -3dt\right) = 0,$$

$$a_n = \int_0^1 3\cos\frac{n\pi t}{2}dt + \int_1^2 -3\cos\frac{n\pi t}{2}dt$$

$$= \frac{6}{n\pi}\sin\frac{n\pi t}{2}\Big|_{t=0}^{1} - \frac{6}{n\pi}\sin\frac{n\pi t}{2}\Big|_{t=1}^{2} = \frac{12}{n\pi}\sin\frac{n\pi}{2}.$$

Hence, $F(t) = \sum_{n=1}^{\infty} \frac{12}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{2}.$

Try with $x(t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi t}{2}$, since $F(t)$ is an infinite sum of cosine series and there is no x' in the equaiton. Also note that $\cos\frac{n\pi t}{2}$ $(n = 1, 2, \ldots)$ does not duplicate with the solution in $x_c = c_1\cos\sqrt{5}\,t + c_2\sin\sqrt{5}\,t$.

$$\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{4}A_n\cos\frac{n\pi t}{2} + 5\sum_{n=1}^{\infty} A_n\cos\frac{n\pi t}{2} = \sum_{n=1}^{\infty}\frac{12}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{2}.$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(5 - \frac{n^2\pi^2}{4}\right)A_n\cos\frac{n\pi t}{2} = \sum_{n=1}^{\infty}\frac{12}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{2}.$$

$$\left(5 - \frac{n^2\pi^2}{4}\right)A_n = \frac{12}{n\pi}\sin\frac{n\pi}{2}$$

$$\Rightarrow A_n = \frac{\frac{12}{n\pi}}{5 - \frac{n^2\pi^2}{4}}\sin\frac{n\pi}{2} = \frac{\frac{48}{n\pi}}{20 - n^2\pi^2}\sin\frac{n\pi}{2} = \frac{48}{n\pi(20 - n^2\pi^2)}\sin\frac{n\pi}{2}.$$

Hence,

$$x(t) = \sum_{n=1}^{\infty}\frac{48}{n\pi(20 - n^2\pi^2)}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{2}.$$

Also note that.

$$\sin\frac{n\pi}{2} = \begin{cases} 1 & n = 4k+1 \\ 0 & n = 4k+2 \\ -1 & n = 4k+3 \\ 0 & n = 4k+4 \end{cases}$$