Section 1.2

#6 $\frac{dy}{dx} = x\sqrt{x^2+9}$; $y(-4)=0$

Sol) $y = \int x\sqrt{x^2+9}\,dx = \frac{1}{3}(x^2+9)^{3/2} + C$

$u = x^2+9$, $du = 2x\,dx$ (substitution method)

$0 = \frac{1}{3}(16+9)^{3/2} + C = \frac{125}{3} + C$

$\Rightarrow C = -\frac{125}{3}$

Hence, $\boxed{y = \frac{1}{3}(x^2+9)^{3/2} - \frac{125}{3}}$

#16 $a(t) = \frac{1}{\sqrt{t+4}}$; $v_0 = -1$, $x_0 = 1$ $\Rightarrow x(t) = ?$

Sol) $v(t) = \int a(t)\,dt = \int \frac{1}{\sqrt{t+4}}\,dt = 2\sqrt{t+4} + C$

$-1 = 2\sqrt{0+4} + C = 4 + C \Rightarrow C = -5$

Hence,

$x(t) = \int v(t)\,dt = \int (2\sqrt{t+4} - 5)\,dt$

$= \frac{4}{3}(t+4)^{3/2} - 5t + C$

$1 = \frac{4}{3}(0+4)^{3/2} + C$, since $x_0 = 1$

$\Rightarrow C = -\frac{29}{3}$

Thus, $\boxed{x(t) = \frac{4}{3}(t+4)^{3/2} - 5t - \frac{29}{3}}$

#24 $a = -g = -32 ft/s^2$, $x(0) = 400$ ft. $v(0) = 0$.

Hence,

$$v(t) = \int -32\,dt = -32t + C$$

$$0 = C, \text{ since } v(0) = 0$$

$$\therefore x(t) = \int v(t)\,dt = \int -32t\,dt = -16t^2 + C.$$

$x(0) = 400 \Rightarrow C = 400$.

$\therefore \quad x(t) = -16t^2 + 400$

$t = ?$ when $x(t) = 0$, i.e., $-16t^2 + 400 = 0$

$$-16t^2 + 400 = 0 \Rightarrow t = 5, \cancel{-5}$$

Hence, $\boxed{v(5) = -32 \cdot 5 = -160 \text{ ft/s}}$

Section 1.3

#13 $f(x,y) = \sqrt[3]{y}$: continuous near $(0,1)$

$\Rightarrow$ existence is guaranteed.

$$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3} = \frac{1}{3} \frac{1}{\sqrt[3]{y^2}} \text{ : continuous near } (0,1)$$

$\Rightarrow$ uniqueness is guaranteed.

Hence, $\boxed{\text{both existence and uniqueness are guaranteed.}}$

#14 $f(x,y) = \sqrt[3]{y}$: continuous near $(0,0)$

$\Rightarrow$ existence is guaranteed.

$\frac{\partial f}{\partial y} = \frac{1}{3}\frac{1}{\sqrt[3]{y^2}}$: not continuous near $(0,0)$

$\Rightarrow$ uniqueness is not guaranteed.

Hence, $\boxed{\text{only existence is guaranteed.}}$

#18 $f(x,y) = \frac{x-1}{y}$: not continuous near $(1,0)$

$\frac{\partial f}{\partial y} = -\frac{(x-1)}{y^2}$: not continuous near $(1,0)$

Hence,

$\boxed{\text{the existence - uniqueness theorem guarantees nothing.}}$

Section 1.4

#4. $(1+x)\frac{dy}{dx} = 4y$

sol)

$$\frac{1}{4y}\,dy = \frac{1}{(1+x)}\,dx$$

$$\int \frac{1}{4y}\,dy = \int \frac{1}{1+x}\,dx$$

$$\frac{1}{4}\ln|y| = \ln|1+x| + c$$

$$\ln|y| = 4\ln|1+x| + D = \ln(1+x)^4 + D$$

$$|y| = e^{\ln(1+x)^4 + D} = e^D \cdot e^{\ln(1+x)^4} = e^D \cdot (1+x)^4.$$

Hence, $\boxed{y = A(1+x)^4, \quad A: \text{arbitrary constant.}}$

#11 $y' = xy^3$

sol) $\frac{dy}{dx} = xy^3$

$\frac{1}{y^3}\,dy = x\,dx$

$\int \frac{1}{y^3}\,dy = \int x\,dx$

$-\frac{1}{2}y^{-2} = \frac{x^2}{2} + C$

$y^{-2} = -x^2 + C$

$y = (-x^2 + C)^{-1/2}$, using C abusively.

#20. $\frac{dy}{dx} = 3x^2(y^2+1)$, $y(0)=1$

()

$\frac{1}{y^2+1}\,dy = 3x^2\,dx$

$\int \frac{1}{y^2+1}\,dy = \int 3x^2\,dx$

$\arctan y = x^3 + C$

$\arctan 1 = C \Rightarrow C = \frac{\pi}{4}$.

$\therefore \arctan y = x^3 + \frac{\pi}{4}$

$\therefore \boxed{y = \tan\left(x^3 + \frac{\pi}{4}\right)}$

26 $\quad \dfrac{dy}{dx} = 2xy^2 + 3x^2y^2 , \qquad y(1) = -1$

(i) $\quad$ Note that $\quad 2xy^2 + 3x^2y^2 = (2x+3x^2)y^2$

$$\frac{dy}{dx} = (2x+3x^2)y$$

$$\frac{1}{y^2}dy = (2x+3x^2)\,dx$$

$$\int \frac{1}{y^2}dy = \int (2x+3x^2)\,dx$$

$$-\frac{1}{y} = x^2 + x^3 + C$$

$$y = -\frac{1}{x^2+x^3+C}$$

$$-1 = -\frac{1}{1+1+C} = -\frac{1}{2+C} \Rightarrow C = -1$$

$$\therefore \boxed{y = -\frac{1}{x^2+x^3-1} \quad \left(= \frac{1}{1-x^2-x^3}\right)}$$

#34. $\frac{dP}{dt} = kP$

$\Rightarrow \frac{1}{P} dP = k\,dt$

$\Rightarrow \ln P = kt + c$

$\Rightarrow P = e^{kt+c} = De^{kt}$

We know that $P(10) = 6P(0) = 6D$

$\therefore \quad 6D = De^{10k}$

$\Rightarrow 6 = e^{10k} \Rightarrow 10k = \ln 6 \Rightarrow k = \frac{\ln 6}{10}$.

$t = ?$ when $P(t) = 2D$.

$2D = De^{\frac{\ln 6}{10} t}$

$\Rightarrow 2 = e^{\frac{\ln 6}{10} \cdot t}$

$\Rightarrow \frac{\ln 6}{10} \cdot t = \ln 2$

$\Rightarrow \boxed{t = \ln 2 \cdot \frac{10}{\ln 6} \approx 3.87 \text{ hrs}}$