Sec 2.1

5. $\frac{dx}{dt} = 3x(5-x), \qquad x(0) = 8$

Sol)

$$\frac{dx}{dt} = -3x(x-5),$$

$$\frac{1}{x(x-5)}\,dx = -3\,dt$$

$$\frac{1}{5}\int\left(\frac{1}{x-5} - \frac{1}{x}\right)dx = \int -3\,dt$$

$$\frac{1}{5}\ln\left|\frac{x-5}{x}\right| = -3t + c$$

$$\frac{x-5}{x} = c\cdot e^{-15t}$$

$$x - 5 = c\cdot e^{-15t}\,x$$

$$x(1 - c\,e^{-15t}) = 5$$

$$x = \frac{5}{1 - c\,e^{-15t}}$$

$x(0) = 8$; $\quad 8 = \frac{5}{1-c} \Rightarrow 8 - 8c = 5$

$$\Rightarrow c = \frac{3}{8}.$$

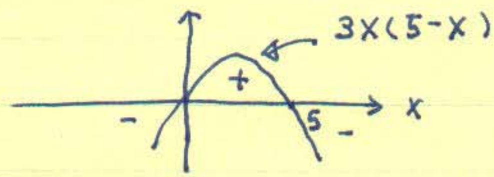
$$\therefore\ x(t) = \frac{5}{1 - \frac{3}{8}e^{-15t}} = \frac{40}{8 - 3e^{-15t}}$$

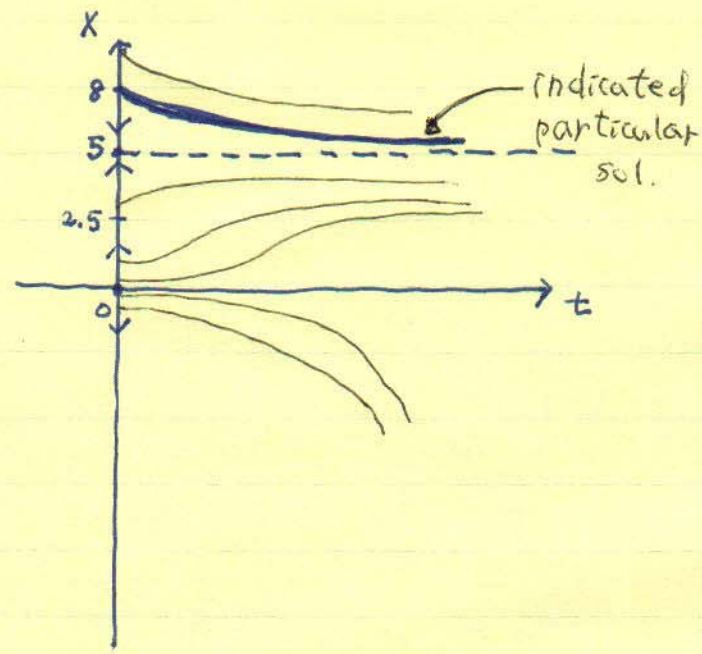
$$\left(= \frac{40e^{15t}}{8e^{15t} - 3}\right)$$

Note that $x(t) \to \frac{40}{8} = 5$ as $t \to \infty$.

Typical Solution Curves :

Critical points : $x = 0, 5$





Note :

$\frac{dx}{dt}$	solution curve
+, increasing	increasing, concave up
+, decreasing	increasing, concave down
−, increasing	decreasing, concave ~~up~~ down
−, decreasing	decreasing, concave up

Sec 2.1

6. $\dfrac{dx}{dt} = 3x(x-5), \quad x(0) = 2$

Sol)

$$\frac{1}{5}\left(\frac{1}{x-5} - \frac{1}{x}\right) dx = 3\,dt$$

$$\int\left(\frac{1}{x-5} - \frac{1}{x}\right) dx = \int 15t$$

$$\ln\left|\frac{x-5}{x}\right| = 15t + C$$

$$\frac{x-5}{x} = c \cdot e^{15t}$$

$$x = \frac{15}{1 - ce^{15t}}$$

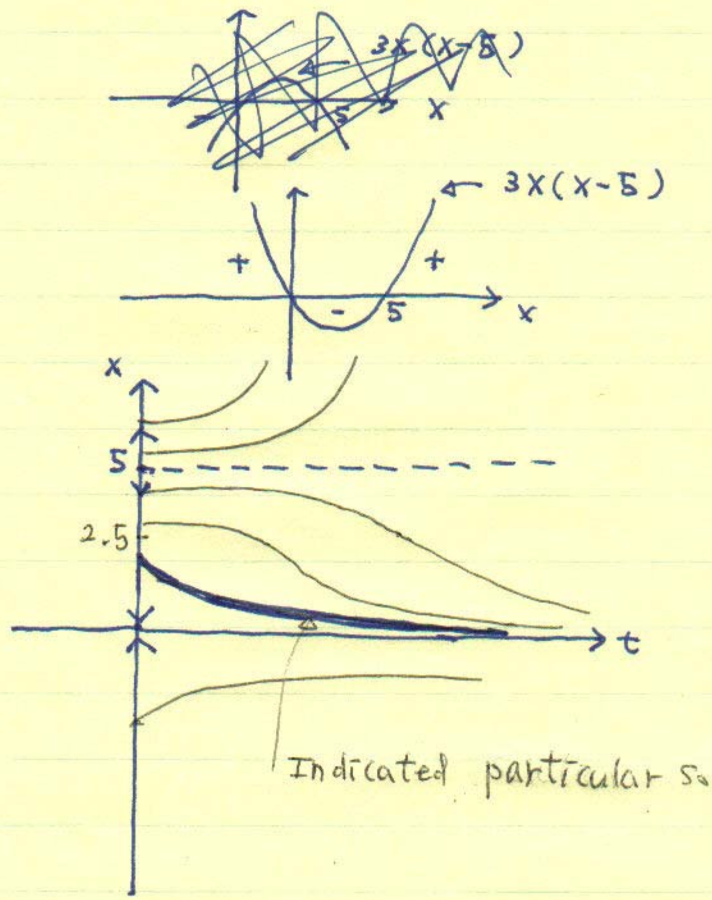
$x(0) = 2$; $\quad 2 = \dfrac{15}{1-c} \Rightarrow c = -\dfrac{13}{2}$

$$\therefore\ x(t) = \frac{5}{1 + \frac{13}{2}e^{15t}} = \frac{10}{2 + 13e^{15t}}$$

Note that $x(t) \to 0$ as $t \to \infty$.

Typical Solution Curves:

critical points: $x = 0, 5$



Sec 3.1

4. $y'' + 25y = 0$; $y_1 = \cos 5x$, $y_2 = \sin 5x$
; $y(0) = 10$, $y'(0) = -10$

Sol)

Verification : routine.

$y = c_1 \cos 5x + c_2 \sin 5x$

$y' = -5c_1 \sin 5x + 5c_2 \cos 5x$

$y(0) = 10$: $c_1 = 10$

$y'(0) = -10$: $5c_2 = -10 \Rightarrow c_2 = -2$

$\therefore \underline{y = 10\cos 5x - 2\sin 5x}$.

---

10. $y'' - 10y' + 25y = 0$; $y_1 = e^{5x}$, $y_2 = xe^{5x}$
; $y(0) = 3$, $y'(0) = 13$

Sol) Verification : omit.

$y = c_1 e^{5x} + c_2 x e^{5x}$

$y' = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x e^{5x}$

$y(0) = 3$: $c_1 = 3$

$y'(0) = 13$: $5c_1 + c_2 = 13 \Rightarrow c_2 = -2$

$\therefore \underline{y = 3e^{5x} - 2xe^{5x}}$

18.

Sol) $y = x^3$, $y' = 3x^2$, $y'' = 6x$

$\therefore$ $yy'' = 6x^4 \Rightarrow y = x^3$: a solution.

Suppose $y = cx^3$ is a solution.

Then $y' = 3cx^2$, $y'' = 6cx$,

$\Rightarrow$

$yy'' = 6c^2x^4 = 6x^4 \Rightarrow \underline{c^2 = 1}$.

Hence, if $c^2 \neq 1$, then $y = cx^3$ is not a sol.

#22. $f(x) = 1 + x$, $g(x) = 1 + |x|$

Sol)

$$g(x) = 1 + |x| = \begin{cases} 1 + x, & \text{if } x \geq 0 \\ 1 - x, & \text{if } x \leq 0 \end{cases}$$

Hence, $\dfrac{f(x)}{g(x)} = 1$ if $x \geq 0$,

but $\dfrac{f(x)}{g(x)} \neq 1$ if $x < 0$.

Thus, there is no constant c s.t

$f(x) = c \cdot g(x)$ for all $x \in \mathbb{R}$.

Thus, they are linearly independent.

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34. $y'' + 2y' - 15y = 0$

Sol) Characteristic eq :

$r^2 + 2r - 15 = 0$

$(r+5)(r-3) = 0 \Rightarrow r = 3, -5$

$\therefore$ Two linearly indept sols are

$y_1 = e^{3x}$, $y_2 = e^{-5x}$.

$\therefore$ a general solution is

$y = c_1 e^{3x} + c_2 e^{-5x}$

---

38. $4y'' + 8y' + 3y = 0$

Sol) Characteristic eq :

$4r^2 + 8r + 3 = (2r+1)(2r+3) = 0$

$r = -\frac{1}{2}, -\frac{3}{2}$.

$\therefore$ a general solution is

$y = c_1 e^{-\frac{1}{2}x} + c_2 e^{-\frac{3}{2}x}$

40. $9y'' - 12y' + 4y = 0.$

sol) Characteristic eq: $9r^2 - 12r + 4 = 0$

$$(3r-2)^2 = 0 \Rightarrow r = \frac{2}{3} \text{ (repeated roots)}$$

Two linearly indept sols are $y_1 = e^{2/3\,x}$, $y_2 = x e^{2/3\,x}$.

Hence, a general solution is

$$y = c_1 e^{2/3\,x} + c_2 x e^{2/3\,x} \quad \left(= (c_1 + c_2 x) e^{2/3\,x} \right)$$