Section 3.4

#2. $mx'' + kx = 0$

$0.75x'' + 48x = 0$

· circular frequency : $\sqrt{\frac{48}{0.75}} = 8$ rad/s

· Period : $\frac{2\pi}{8} = \frac{\pi}{4}$ sec

· frequency $= \frac{1}{\text{period}} = \frac{4}{\pi}$ /sec

#4a) $\begin{cases} m = 250g = 0.25kg \\ k = \frac{9N}{25cm} = \frac{9N}{0.25m} = 36\ N/m \\ x(0) = 1\ m, \quad x'(0) = -5\ m/s. \end{cases}$

$mx'' + kx = 0$

$0.25x'' + 36x = 0$

Char. eq : $0.25r^2 + 36 = 0$

$r = \pm\sqrt{\frac{36}{0.25}}\ i = \pm 12i$

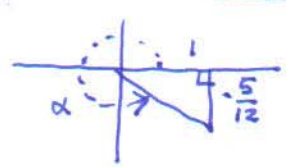
$\therefore x(t) = c_1 \cos 12t + c_2 \sin 12t$

$x'(t) = -12c_1 \sin 12t + 12c_2 \cos 12t.$

$x(0) = 1$; $c_1 = 1$

$x'(0) = -5$; $12c_2 = -5 \Rightarrow c_2 = -\frac{5}{12}.$

$$\boxed{\begin{aligned} \therefore x(t) &= \cos 12t - \frac{5}{12}\sin 12t \\ &= \sqrt{1 + \frac{25}{144}}\ \cos(12t - \alpha). \\ &= \frac{13}{12}\cos(12t - \alpha), \end{aligned}}$$



where

$\alpha = 2\pi - \tan\left(\frac{5}{12}\right)$

$\cong 5.8884.$

4b) Amplitude $= \frac{13}{12}$

period $= \frac{2\pi}{12} = \frac{\pi}{6}.$

#5. $L\theta'' + g\theta = 0$

Period

$P = \frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi\frac{\sqrt{L}}{\sqrt{g}}.$

Hence,

$P_1 = 2\pi\frac{\sqrt{L_1}}{\sqrt{\frac{GM}{R_1^2}}} = \frac{2\pi}{\sqrt{GM}} R_1\sqrt{L_1}$

$P_2 = 2\pi\frac{\sqrt{L_2}}{\sqrt{\frac{GM}{R_2^2}}} = \frac{2\pi}{\sqrt{GM}} R_2\sqrt{L_2}$

$\therefore \frac{P_1}{P_2} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}.$

#16. $3x'' + 30x' + 63x = 0$

$D = (30)^2 - 4\cdot 3\cdot 63 > 0$

i.e. $(30)^2 > 4\cdot 3\cdot 63 = 756$

$\Rightarrow$ overdamped motion

Char. eq: $3r^2 + 30r + 63 = 0$

$r^2 + 10r + 21 = 0$

$(r+3)(r+7) = 0$

$r = -3, -7$

$x(t) = c_1 e^{-3t} + c_2 e^{-7t}$

$x'(t) = -3c_1 e^{-3t} - 7c_2 e^{-7t}$

$$\left.\begin{array}{ll} x(0) = 2 \ ; & c_1 + c_2 = 2 \\ x'(0) = 2 \ ; & -3c_1 - 7c_2 = 2 \end{array}\right\}$$

$\Rightarrow c_2 = -2, \ c_1 = 4$

$$\boxed{\therefore \ x(t) = 4e^{-3t} - 2e^{-7t}.}$$

#18. $mx'' + cx' + kx = 0$

$2x'' + 12x' + 50x = 0$

$D = (12)^2 - 4\cdot 2\cdot 50 < 0$

$\Rightarrow$ underdamped motion.

Chat. eq: $2r^2 + 12r + 50 = 0$

$r^2 + 6r + 25 = 0$

$r = -3 \pm \sqrt{9-25} = -3 \pm 4i$

$\therefore \ x(t) = e^{-3t}(c_1 \cos 4t + c_2 \sin 4t)$

$x'(t) = -3e^{-3t}(c_1 \cos 4t + c_2 \sin 4t)$
$\quad + e^{-3t}(-4c_1 \sin 4t + 4c_2 \cos 4t)$

$x(0) = 0 \ ; \quad c_1 = 0$

$x'(0) = -8 \ ; \quad 4c_2 = -8 \Rightarrow c_2 = -2.$

Hence,

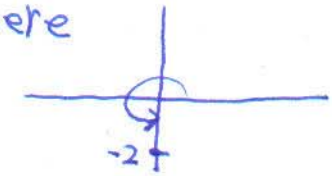
$x(t) = -2e^{-3t} \sin 4t$

$= e^{-3t}(-2\sin 4t)$

$= \sqrt{0+4}\, e^{-3t} \cos(4t - \alpha)$

$= 2e^{-3t} \cos(4t - \alpha),$

where



$\alpha = \frac{3\pi}{2}$

$$\therefore \boxed{x(t) = 2e^{-3t} \cos\left(4t - \frac{3\pi}{2}\right)}$$

Section 3.5

#2. $y'' - y' - 2y = 3x + 4$

Sol) Char. eq: $r^2 - r - 2 = (r-2)(r+1) = 0$

$r = 2, -1 \Rightarrow e^{2x}, e^{-x}$

Try $y_p = Ax + B$ ($\Rightarrow$ No duplication in y_c)

$y_p' = A, \quad y_p'' = 0$

$\therefore y_p'' - y_p' - 2y_p$

$= -2Ax - (2B + A) = 3x + 4$

$\Rightarrow -2A = 3 \Rightarrow A = -\frac{3}{2}$

$2B + A = -4 \Rightarrow B = -\frac{5}{4}$

$\therefore \boxed{y_p = -\frac{3}{2}x - \frac{5}{4}.}$

#4. $4y'' + 4y' + y = 3xe^x$

Sol) Char. eq: $4r^2 + 4r + 1$

$= (2r+1)^2 = 0$

$r = -\frac{1}{2}, -\frac{1}{2} \Rightarrow e^{-\frac{1}{2}x}, xe^{-\frac{1}{2}x}$

Try $y_p = (Ax + B)e^x$ ($\Rightarrow$ No duplication)

$y_p' = Ae^x + (Ax+B)e^x = (A+B)e^x + Axe^x$

$y_p'' = 2Ae^x + (Ax+B)e^x$

$= (2A+B)e^x + Axe^x$

$\therefore 4y_p'' + 4y_p' + y_p$

$= (8A + 4B + 4A + 4B + B)e^x$

$+ (4A + 4A + A)xe^x = 3xe^x$

$\Rightarrow (12A + 9B)e^x + 9Axe^x = 3xe^x$

$\Rightarrow 12A + 9B = 0$

$9A = 3$

$\Rightarrow A = \frac{1}{3}, \quad B = -\frac{4}{9}.$

Hence,

$\boxed{y_p = \left(\frac{1}{3}x - \frac{4}{9}\right)e^x.}$

#10. $y'' + 9y = 2\cos 3x + 3\sin 3x$

Sol) Char. eq: $r^2 + 9 = 0 \Rightarrow r = \pm 3i$

$\Rightarrow \cos 3x, \sin 3x$

~~duplication~~

$y_p = x(A\cos 3x + B\sin 3x)$

multiply by x to eliminate duplication

$y_p'' = 2(-3A\sin 3x + 3B\cos 3x)$

$+ x(-9A\cos 3x - 9B\sin 3x)$

$\therefore y_p'' + 9y_p$

$= 6B\cos 3x - 6A\sin 3x$

$= 2\cos 3x + 3\sin 3x$

$\Rightarrow \begin{cases} 6B = 2 \\ -6A = 3 \end{cases} \Rightarrow \begin{cases} B = \frac{1}{3} \\ A = -\frac{1}{2} \end{cases}$

Hence,

$\boxed{y_p = x\left(-\frac{1}{2}\cos 3x + \frac{1}{3}\sin 3x\right)}$