Observation: The Eisenstein series $g_n = 2 \cdot G_n$ satisfy all these conditions.

At $p=2$, the story is not as clean ...

$$\pi_q \operatorname{fibre}(gl_1 \, tmf \longrightarrow L_2 \, gl_1 \, tmf) = 0 \text{ if } q>3$$

More generally, if $R = L_n R$ and R is E_∞ ring spectrum, then

$$\pi_q \operatorname{fibre}(gl_1 R \longrightarrow L_n \, gl_1 R) = 0 \text{ if } q > n+1.$$

The key input is:

$$K(n)_* \, K(q, \mathbb{Z}/p) = 0 \quad \text{if } q > n.$$

17.10.03

$L_{K(2)} \, tmf$

Let R be an $\mathbb{F}_p$-algebra, C/R elliptic curve.

C is super-singular if $\hat{C}$ has strict height 2.

Example: $y^2 + y = x^3 / \mathbb{F}_2$

Classical: Over a field $R = k$, elliptic curves are classified by $j(C) \in k$ if $k = \bar{k}$ is algebraically closed.

Super-singular curves correspond to a finite set of j-values.

$\mathcal{M}_{ss}(R) =$ groupoid of super-singular curves over R

$=$ the locus where $v_1 = 0$

$$v_1 \in \Gamma(\mathcal{M}_{ell} \otimes \mathbb{F}_p, \omega^{\otimes p-1})$$

At primes ≥ 11, there is more than one iso class of s.s. elliptic curves, so then $L_{K(2)}tmf$ is a product of homotopy fixed point spectra.

<u>Fact</u>: for $p \geq 5$, $p \nmid \# G_i$.

<u>Where is the elliptic curve?</u>

Serre-Tate deformation theory for abelian varieties:

A ring, I ideal nilpotent, $p \in I$,

$A_0 = A/I$

$$\begin{tikzcd}
\mathcal{M}_{ell}(A) \arrow[rr, "\text{reduction}"] \arrow[dr, "\simeq"'] & & \mathcal{M}_{ell}(A_0) \\
& \mathrm{Def}(A, A_0) &
\end{tikzcd}$$

objects:
- C_0/A_0 elliptic curve
- p-divisible group G/A
- isomorphism $\varepsilon : G_0 \xrightarrow{\cong} C_0[p^\infty]$

Scheme-theoretic pullback

$$\begin{tikzcd}
C[p^h] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_0 \arrow[d, "{[p^h]}"] \\
\mathrm{Spec}(A_0) \arrow[r] & C_0
\end{tikzcd}$$

p-divisible group = inductive limit of such group schemes.

$$C_0[p^h]^1 \rightarrowtail C_0[p^h] \twoheadrightarrow C_0[p^h]^{et}$$

formal part

étale part (= 0 for strict height 2)

Thus if C_0 is super-singular, then all we need to get a curve on A is a lift of the formal group. Lubin-Tate theory does this, and so there is a unique lift.

$\Rightarrow$ "Near the locus of super-singular curves, the morphism $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is étale"