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The Witten genus

M^k Riemannian Spin manifold, D = Dirac operator, T = tangent bundle

Witten genus

$$w(M) = \operatorname{ind}\Big(D \otimes \bigotimes_{n \geq 1} S_{q^n}(T^{\mathbb{C}} - \underline{r}) \Big)$$

where

$$S_t(V) = \sum_{k \geq 0} t^k \cdot S^k(V) \qquad \in \mathbb{Z}[[q]]$$

note: $\quad S_t(V)^{-1} = \Lambda_{-t}(V)$

Characteristic series

$$\sigma(L,q) = (L^{1/2} - L^{-1/2}) \prod_{n \geq 1} \frac{(1-q^n L)(1-q^n L^{-1})}{(1-q^n)^2}$$

For $\left.\begin{array}{l} u = e^{2\pi i z} \\ q = e^{2\pi i \tau} \end{array}\right\}$ $\quad \sigma(u,q) = (u^{1/2} - u^{-1/2}) \prod_{n \geq 1} \frac{(1-q^n u)(1-q^n u^{-1})}{(1-q^n)^2}$

is essentially the Weierstrass σ-function for the lattice $\Lambda = 2\pi i \mathbb{Z} + 2\pi i \tau \mathbb{Z}$. I.e. σ defines a holomorphic function of $z \in \mathbb{C}$ which vanishes exactly on Λ.

Given points on the elliptic curve $C = \mathbb{C}/\Lambda$.

$P_1, \dots, P_n, Q_1, \dots, Q_n \in C$ represented by

$\tilde{P}_1, \dots, \tilde{P}_n, \tilde{Q}_1, \dots, \tilde{Q}_n \in \mathbb{C}$.

Suppose $\quad 0 = \sum P_i - \sum Q_i \iff \sum \tilde{P}_i - \sum \tilde{Q}_i \in \Lambda$

Then there exists a function, unique up to scalar multiple, f on C with divisor

$\sum (P_i) - \sum (Q_i)$, namely

$$\prod_{i=1}^{n} \frac{\sigma(z - \tilde{P}_i)}{\sigma(z - \tilde{Q}_i)}$$

Thm (Witten, Zagier): If $\frac{p_1}{2}(M)=0$, then $w(M)$ is the q-expansion of a modular form for $SL_2(\mathbb{Z})$.

Def An *elliptic spectrum* consists of

E : commutative ring spectrum, even periodic

C : elliptic curve over $\pi_0 E$

Then $E^0(\mathbb{CP}^\infty)$ is the ring of functions on a formal group G_E over $\pi_0 E$

$t : G_E \cong \hat{C}$ an iso of formal groups over $\pi_0 E$.

Thm (Ando-Hopkins-Strickland) If (E, C, t) is an elliptic spectrum, then there is a canonical map

$$\sigma(E,C,t) : MU\langle 6\rangle \longrightarrow E$$

natural in (E,C,t).

Example: Tate elliptic spectrum

$$\begin{tikzcd} \hat{\mathbb{G}}_m \arrow[r, phantom, "\underset{can.}{\cong}"] & \widehat{\mathrm{Tate}} \arrow[r] \arrow[d] & \mathrm{Tate} \arrow[dl] \\ & \mathbb{Z}[[q]] & \end{tikzcd}$$

$\mathrm{Tate} = \text{Tate curve}/\mathbb{Z}[[q]]$

So get elliptic spectrum $K_{Tate} = (K[[q]], \mathrm{Tate}, \mathrm{can.})$.

Moreover, the diagram commutes

$$\begin{tikzcd} MU\langle 6\rangle \arrow[r, "\sigma(K_{Tate})"] \arrow[d] & K[[q]] \\ MSU \arrow[r] & MSpin \arrow[u, "w"'] \end{tikzcd}$$